Oscillations, Waves & Optics

for

M.Sc. Entrance

IIT-JAM | JNU | TIFR | BHU | DU

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PREFACE

The present booklet on Oscillations, Waves & Optics section of the Physics-PH deals with the syllabi mentioned in the IIT-JAM & other M.Sc. Entrance Exams in their respective notificaions. The Booklet has been designed and organised in a manner so that the aspirants can make full use of it for their immaculate preparation.

The panel of experts and faculties in the R&D team of Career Endeavour have done lots of hardwork and that too with immense care in formulating the booklet into its present form. A brief discussion is given on every topic mentioned in syllabus alongwith ample of solved examples on the concepts discussed. The idea is not only to bring out the basic concepts clearly but also to help the students in developing the analytical approach for solving any problem.

This booklet contains ***Previous Year Questions*** *of IIT-JEE, IIT-JAM, IIT-JAM(GP), JNU, TIFR, JEST and IISc as solved examples relevant to the topic. It is expected that the booklet will form an integral part of the Classroom/ Corresspondence program which the student would join for his/her success at Career Endeavour. Even though due care has been taken while preparing the booklet, the booklet may contain errors/mistakes here and there, we sincerely apologize for the same.We would be also thankful to those readers who point out the errors.*

Finally, we from the R&D team of Career Endeavour, wish all the aspirants 'GOOD LUCK' and will always be available to help them out so that they come up with 'flying colors'.

CONTENT

Chapter **Page Nos.**

CHAPTER-1
Simple Harmonic Motion ..... (1-23)

CHAPTER-2
Damped & Force Vibrations ..... (24-42)

CHAPTER-3
Acoustic Wave Motion ..... (43-58)

CHAPTER-4
Phase & Group Velocity ..... (59-63)

CHAPTER-5
Doppler Effect ..... (64-74)

CHAPTER-6
Fermat's Principle ..... (75-81)

CHAPTER-7
Refraction at Spherical Surfaces ..... (82-100)

CHAPTER-8
Thick Lens & Lens combinations ..... (101-109)

CHAPTER-9
Interference of Light ..... (110-143)

CHAPTER-10
Diffraction of Light ..... (144-161)

CHAPTER-11
Polarisation of light ..... (162-183)

Reference Books ..... (184)

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Chapter 1

Simple Harmonic Motion

1.1 DESCRIPTION OF SIMPLE HARMONIC MOTION (SHM):

1.1.1 Equation of motion:

Suppose a particle of mass m is undergoing SHM along a line. If x be its displacement at any instant from the position of stable equilibrium, then restoring force F under small oscillation approximation may be written as

$$F = -sx \qquad \ldots (1)$$

where s is the restoring force per unit displacement, called force constant.
The negative sign indicates that F and x are oppositely directed. For spring restoring force follows Hooke's law of elasticity. From Newton's law, the equation of motion is

$$m\frac{d^2x}{dt^2} = -sx \Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad \ldots (2)$$

where $\omega^2 = s/m$.
Equation (2) is the differential equation of motion of a simple harmonic oscillator.
Its general solution is given by the principle of superposition as

$$x = c_1 e^{j\omega t} + c_2 e^{-j\omega t} \qquad \ldots (3)$$

$$\Rightarrow \quad x = c_1(\cos\omega t + j\sin\omega t) + c_2(\cos\omega t - j\sin\omega t)$$

$$= A\cos\omega t + B\sin\omega t \qquad \ldots (4)$$

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where, $A = (c_1 + c_2)$ and $B = j(c_1 - c_2)$.
Again, introducing two other constants a and ε by the relations $A = a\sin\varepsilon$ and $B = a\cos\varepsilon$, such that $a = \sqrt{A^2 + B^2}$ and $\tan\varepsilon = A/B$, solution (4) can be put in the form,

$$x = a\sin(\omega t + \varepsilon) \qquad \ldots (5)$$

Putting $\varepsilon = \frac{\pi}{2} + \delta$, the solution (5) can also be expressed as

$$x = a\cos(\omega t + \delta) \qquad \ldots (6)$$

1.1.2 Characteristics of SHM:

Let us choose the solution, $x = a\cos(\omega t + \delta)$

Here *'a'* represents the maximum value of displacement from the position of stable equilibrium. It is called the **amplitude** of SHM. The **time period** T is the time interval in which the motion repeats itself and is given by

$$x(t) = x(t+T)$$

$$\Rightarrow \quad a\cos(\omega t + \delta) = a\cos[\omega(t+T)+\delta] = a\cos[\omega t + \delta + \omega T]$$

$$\Rightarrow \quad a\cos[(\omega t + \delta)+2\pi] = a\cos[(\omega t + \delta) + \omega T]$$

For equality we must have, $\omega T = 2\pi$ or, $T = 2\pi/\omega = 2\pi\sqrt{\frac{m}{s}}$ ... (7)

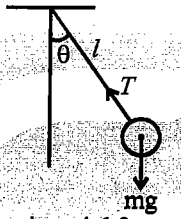
The quantity ω is called the **angular frequency**. The usual frequency ν which gives the number of oscillations per unit time is given by $\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$ ... (8)

The quantity $(\omega t + \delta)$ in equation (6) is called the phase of motion and δ is called the initial phase or epoch.

Effective acceleration due to gravity (g')

When simple pendulum is kept in inertial frame its time period is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$



When simple pendulum is kept in non-inertial frame its time period is given by

$$T = 2\pi\sqrt{\frac{l}{g'}},$$

where g' is called effective gravity and is given by

$$\vec{g}' = \vec{g} - \vec{a} \quad \text{... (9)}$$

where $\vec{a}$ is acceleration of the frame.

Ex. 1. A simple pendulum, suspended from the ceiling of a stationary cart, has a time period 2 seconds. When the cart accelerates in the horizontal direction with an acceleration of 10 m/s, the time period of the pendulum is (g = 10 m/s^{-2})

(a) $2^{1/2}$ seconds (b) $2^{3/2}$ seconds (c) $2^{1/4}$ seconds (d) $2^{3/4}$ seconds

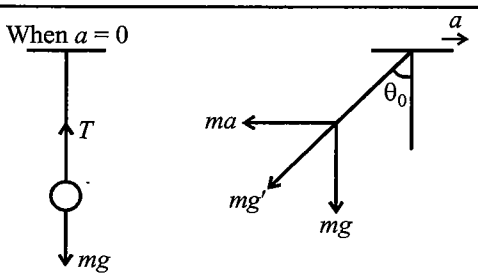
[JAM-GP: 2007]

Ans. (d)

Soln. Forces on bob

1. Gravity downward : mg
2. Tension force : T
3. Psuedo force : $-ma$

Now, at equilibrium condition, sum of gravitational force and pseudo force will be balanced by tension force.



Mean position
$\theta = 0°$

Mean position
$\theta = \theta_0$

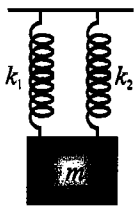
Here $g' = \vec{g} - \vec{a}$

$\Rightarrow g' = \sqrt{a^2 + g^2}$

$$\text{Time period} = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{\sqrt{g^2+a^2}}} = 2\pi\left[\sqrt{\frac{l}{g\sqrt{1+\frac{a^2}{g^2}}}}\right] = 2\pi\sqrt{\frac{l}{g}}\frac{1}{\left(1+\frac{a^2}{g^2}\right)^{1/4}} = 2\times\frac{1}{2^{1/4}} = 2^{3/4}\text{ sec.}$$

Ex. 2. Two springs of force constant k_1 and k_2 are arranged in a parallel arrangement and a mass 'm' is suspended from it. The arrangement is equivalent to a single spring of constant k given by **[DU 2015]**

(a) $k_1 - k_2$ (b) $k_1 + k_2$ (c) $k_1k_2/(k_1+k_2)$ (d) k_1/k_2



Soln. **(b)** If an elongation of x is given to the system, then restoring force

$$F = -k_1x - k_2x = -(k_1+k_2)x = -kx$$

where, $k = k_1 + k_2 \equiv$ equivalent spring constant

Ex. 3. Find the time period for the arrangement of the spring mass system having constant k_1 and k_2.

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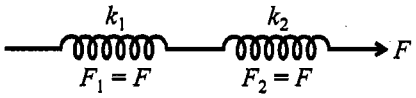
Soln. Let x be the displacement of mass m from its equilibrium position at an instant and x_1 and x_2 are the extension in the springs k_1 and k_2 respectively.

$x = x_1 + x_2$ ... (i)

Tension force at each points in a spring is equal to the external force applied at one of its end,



i.e., $F = F_1 = F_2$ ... (ii)

$-k'x = -k_1x_1 = -k_2x_2$

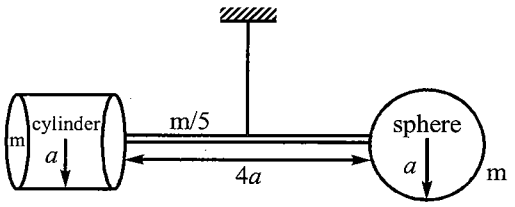
where k' is equivalent spring constant,

From equation (i)

$$\frac{F}{k'} = \frac{F}{k_1} + \frac{F}{k_2} \Rightarrow \boxed{\frac{1}{k'} = \frac{1}{k_1} + \frac{1}{k_2}} \quad \text{... (iii)}$$

i.e., if there is same tension force in all connected springs, equivalent spring constant is given by equation (iii).

Ex. 4. A solid cylinder of mass m, radius a and height also a and a solid sphere of the same mass and the same radius are connected to the two ends of a thin rod of mass m/5 and length $4a$. The line joining the centres of mass of the cylinder and the sphere coincides with the axis of the rod. The vertical cross-section of the assembly is shown in the figure below. The whole assembly is suspended vertically by a massless wire passing through its centre of mass. The torsional constant of the wire is 76 ma^2/sec^2. Calculate the frequency of the torsional oscillation about the suspension in the horizontal plane.



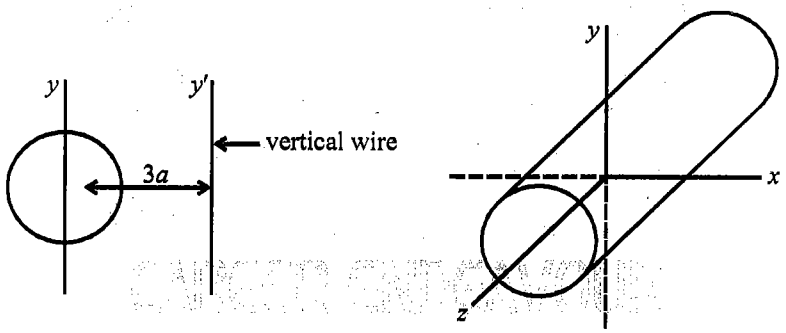
[JAM-GP: 2010]

Soln. Let us calculate the moment of inertia about of cylinder and sphere, and the rod about vertical wire.

Cylinder : $I_z = I_x + I_y = \frac{1}{2}ma^2$ (Perpendicular Axes Theorem)

$$I_x = I_y = \frac{1}{4}ma^2$$

Applying Parallel Axes Theorem, moment of inertia about vertical wire of:



Cylinder : $I_c' = I_{cm} + m(3a)^2 = \frac{1}{4}ma^2 + 9ma^2 = \frac{37}{4}ma^2.$

Sphere : $I_s = I_{cm} + m.(3a)^2 = \frac{2}{5}ma^2 + 9ma^2 = \frac{47}{5}ma^2.$

Rod : $I_r = \frac{1}{12}ML^2 = \frac{1}{12}\cdot\frac{m}{5}\times 16a^2 = \frac{4}{15}ma^2.$

Moment of inertia of system about the vertical wire,

I = sum of moment of inertia of individual bodies about vertical wire

$$= \frac{37}{4}ma^2 + \frac{47}{5}ma^2 + \frac{4}{15}ma^2$$

$$I = \frac{227}{12}ma^2$$

Now, if the rod along with the cylinder and sphere is twisted by an angle of θ, a restoring torsional torque, $\tau = -k\theta$ acts and tends to brings the system back to its original position.

$\tau = -k\theta$

$$I\frac{d^2\theta}{dt^2} = -k\theta \Rightarrow I\frac{d^2\theta}{dt^2} + k\theta = 0$$

$$\frac{227}{12}ma^2\frac{d^2\theta}{dt^2} + 76ma^2\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \frac{912}{227}\theta = 0$$

Angular frequency of torsional oscialltion $\omega = \sqrt{\frac{912}{227}}$ $\quad f = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{912}{227}}$.

1.1.3 Energy in SHM:

Kinetic energy at any instant is given by $E_k = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$

$$\Rightarrow \quad E_k = \frac{1}{2}ma^2\omega^2\sin^2(\omega t + \delta) \qquad \ldots (10)$$

Total potential energy acquired by the particle for the displacement x is

$$E_p = -\int F \cdot dx = \int_0^x sx\,dx = \frac{1}{2}sx^2$$

$$\Rightarrow E_p = \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2a^2\cos^2(\omega t + \delta) \qquad \ldots (11)$$

Total instantaneous energy, $E = E_k + E_p = \frac{1}{2}m\omega^2a^2 = \text{constant}$ ... (12)

and it is equal to the maximum value of the kinetic energy or the potential energy. Thus in SHM energy oscillates between the kinetic and potential forms but total energy remains constant.

Energy position graph:

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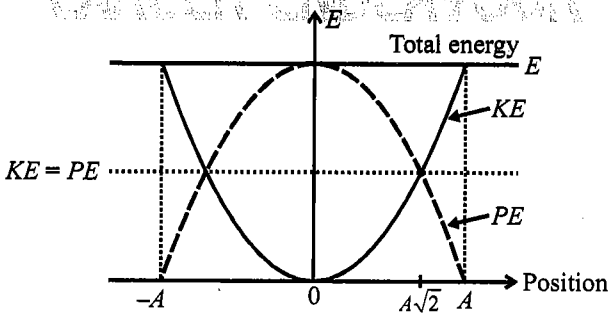

PE : Graph is parabola with vertex at $x = 0$.

KE : The KE curve is inverted parabola.

x. 5. Find the position in terms of amplitude when kinetic energy is equal to potential energy of a particle is SHM.

ns. If $x = A\sin\omega t$ ... (i)

then $v = \frac{dx}{dt} = A\omega\cos\omega t$

$$= A\omega\left[1-\left(\frac{x}{A}\right)^2\right]^{1/2}$$

$$= \omega\left[A^2 - x^2\right]^{1/2} \quad \text{... (ii)}$$

Kinetic energy, $KE = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2\left[A^2 - x^2\right]$ ... (iii)

Potential energy, $PE = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$,

when, $PE = KE$

$$\frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2\left[A^2 - x^2\right]$$

$$\Rightarrow \quad 2x^2 = A^2 \quad \Rightarrow \quad x = \pm\frac{A}{\sqrt{2}}.$$

Ex. 6. Frequency of energy change ω from *KE* to *PE* in SHM is related with the frequency ω_0 of SHM by :

(a) $\omega = 2\omega_0$ (b) $\omega = \frac{\omega_0}{2}$ (c) $\omega = \omega_0^2$ (d) $\omega = \omega_0$

Soln. Potential energy $PE = \frac{1}{2}m\omega_0^2 x^2$

If $x = A\sin\omega_0 t$ then

$$PE = \frac{1}{2}m\omega^2 A^2 \sin^2\omega_0 t = \frac{1}{2}m\omega_0^2 A^2\left[\frac{1-\cos 2\omega_0 t}{2}\right]$$

$$= \frac{1}{4}m\omega_0^2 A^2 - \frac{1}{4}m\omega_0^2 A^2\cos 2\omega_0 t = c - c\cos 2\omega_0 t, \text{ where } c = \frac{1}{4}m\omega_0^2 A^2 = \text{constant}$$

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$$= c - c\cos\omega_E t$$

$\Rightarrow$ frequency of energy change $\boxed{\omega_E = 2\omega_0}$.

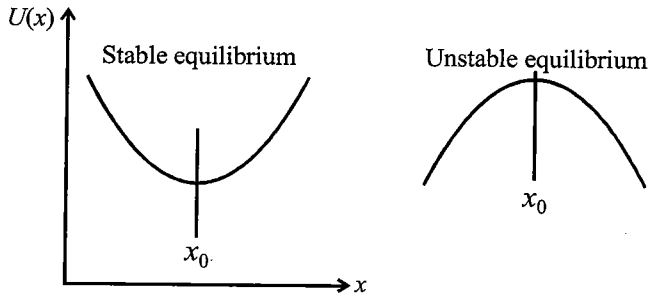
Force Constant in terms of potential energy:

If $U(x)$ is potential energy of the oscillating system in SHM, then at equilibrium point $x = x_0$,

$$F = -\left.\frac{\partial U(x)}{\partial x}\right|_{x_0} = 0.$$

Solving this, we get mean position x_0. The force constant of the oscillating system can be given by

$$K = \left.\frac{d^2U(x)}{dx^2}\right|_{x_0}$$



Ex. 7. A particle of mass 'm' is moving in a potential

$$V(x)=\frac{1}{2}m\omega_0^2x^2+\frac{a}{2mx^2}$$

where ω_0 and 'a' are positive constants. The angular frequency of small oscillations for the simple harmonic motion of the particle about a stable minimum of the potential V(x) is:

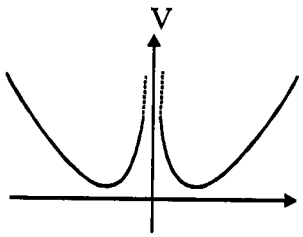
(a) $\sqrt{2}\omega_0$ (b) $2\omega_0$ (c) $4\omega_0$ (d) $4\sqrt{2}\omega_0$

[IIT-JAM : 2011]

Soln. **(b)** $V(x)=\frac{1}{2}m\omega_0^2x^2+\frac{a}{2mx^2}$; $\frac{\partial V}{\partial x}=m\omega_0^2x-\frac{a}{mx^3}$; $\frac{\partial^2V}{\partial x^2}=m\omega_0^2+\frac{3a}{mx^4}$

At minimum potential $\frac{\partial V}{\partial x}=0\,(\text{at } x=x_0)$

$m\omega_0^2x_0=\frac{a}{mx_0^3}\Rightarrow x_0=\left(\frac{a}{m^2\omega_0^2}\right)^{1/4}$, equilibrium position



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Force constant, $k=\left.\frac{\partial^2V}{\partial x^2}\right|_{x=x_0}=m\omega_0^2+\frac{3a}{mx_0^4}=m\omega_0^2+\frac{3m^2\omega_0^2x_0^4}{mx_0^4}=4m\omega_0^2$

$\therefore\quad \omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{4m\omega_0^2}{m}}\Rightarrow \boxed{\omega=2\omega_0}$

1.1.4 Equation of motion from principle of conservation of energy:

Total energy of the oscillator is $\frac{1}{2}m\left(\frac{dx}{dt}\right)^2+\frac{1}{2}sx^2=E\,(\text{constant})$ ... (13)

$\Rightarrow\quad m\frac{dx}{dt}.\frac{d^2x}{dt^2}+sx\frac{dx}{dt}=0\qquad\Rightarrow\left[m\frac{d^2x}{dt^2}+sx\right]\frac{dx}{dt}=0$

Since dx/dt cannot be zero for all values of t, $m\frac{d^2x}{dt^2}+sx=0$ ... (14)

which is the equation of motion.

1. 2 REPRESENTATION OF SHM BY ROTATING VECTOR:

$\overrightarrow{OB}$ is a vector of constant magnitude a which rotates with constant angular velocity ω in anticlockwise sense. At time t = 0 the vector coincides with the line OA which makes an angle δ with x - axis. The x–component of the rotating vector is $x = a\cos(\omega t + \delta)$

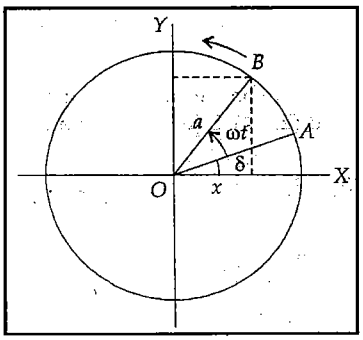


Figure 1.1

which represents a SHM. The y-component of the rotating vector is $y = a\sin(\omega t + \delta)$ which also represents a SHM. From the above treatment it is clear that SHM can be viewed as a projection of circular motion on any fixed diameter of the circle.

1.3 SUPERPOSITION OF COLLINEAR SHMs:

The principle of superposition according to which resultant displacement due to a number of sources is given by the algebraic sum of the displacements caused by the individual sources.

1.3.1 SHMs of same frequency acting along the same direction but having different amplitudes and phases:

Let two SHMs be represented by

$$x_1 = a_1\cos(\omega t + \delta_1) \qquad \ldots (15)$$

$$x_2 = a_2\cos(\omega t + \delta_2) \qquad \ldots (16)$$

where a_1 and a_2 are the amplitudes, δ_1 and δ_2 are the initial phase angles of the two SHMs of same angular frequency ω.

By the superposition principle the resultant displacement is given by

$$x = x_1 + x_2 = a_1\cos(\omega t + \delta_1) + a_2\cos(\omega t + \delta_2)$$

$$= (a_1\cos\delta_1 + a_2\cos\delta_2)\cos\omega t - (a_1\sin\delta_1 + a_2\sin\delta_2)\sin\omega t$$

Putting $a_1\cos\delta_1 + a_2\cos\delta_2 = A\cos\phi$

$$a_1\sin\delta_1 + a_2\sin\delta_2 = A\sin\phi$$

we get $x = A\cos(\omega t + \phi)$ ... (16)

It shows that the resultant motion is also simple harmonic, amplitude of which is given by

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\delta_1 - \delta_2) \qquad \ldots (17)$$

And $$\tan\phi = \frac{a_1\sin\delta_1 + a_2\sin\delta_2}{a_1\cos\delta_1 + a_2\cos\delta_2} \qquad \ldots (18)$$

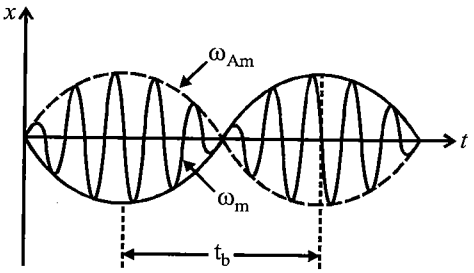
1.3.2 Two SHMs of slightly different frequencies acting along the same direction:

Let us consider two SHMs having slightly different angular frequencies ω_1 and ω_2.

$x_1 = A_0\sin\omega_1 t$

$x_2 = A_0 \sin \omega_2 t$

$$x = x_1 + x_2 = 2A_0 \left[\sin\left(\frac{\omega_1 + \omega_2}{2}\right) t \cos\left(\frac{\omega_1 - \omega_2}{2}\right) t \right]$$



Let $\frac{\omega_1 + \omega_2}{2} = \omega_m$ and $\frac{\omega_1 - \omega_2}{2} = \omega_{Am}$

$$\boxed{x = 2A_0 \sin \omega_m t \cos \omega_{Am} t}$$

$x = (2A_0 \cos \omega_{Am} t) \sin \omega t$

$x = A \cos \omega_m t$

where $A = (2A_0 \cos \omega_{Am} t)$, if $\omega_1 \sim \omega_2$, $\omega_m \to 0$

amplitude $A = 2A_0 \to$ nearly constant.

Then the result oscillation is nearly SHM with frequency ω_{Am} and amplitude $2A_0$.

For maximum amplitude

$A_{max} \Rightarrow \cos \omega_m t = 1$

$\Rightarrow \omega_m t = 2n\pi$

$$\Rightarrow t = \frac{2n\pi}{(\omega_2 - \omega_1)} = \frac{2n\pi}{2\pi(\nu_2 - \nu_1)} = \frac{n}{\nu_2 - \nu_1} = 0, \frac{1}{\nu_2 - \nu_1}, \frac{2}{\nu_2 - \nu_1}, \frac{3}{\nu_2 - \nu_1}, \ldots\ldots$$

time difference between subsequent maxima, $\Delta t = \frac{1}{\nu_2 - \nu_1}$.

Similarly,

time difference between subsequent minima, $\Delta t = \frac{1}{\nu_2 - \nu_1}$.

Beat : One maximum amplitude followed by a minimum, is called a Beat.

Beat time : $t_b = \frac{1}{\nu_2 - \nu_1}$

Beat frequency : $\frac{1}{t_b} = |\nu_2 - \nu_1|$

Ex. 8. The superposition of two harmonic oscillator oscillating in same direction result motion of a point given by $x = a \cos 2.1t \cos 50t$

Calculate (i) individual frequency, (ii) Beat frequency

Soln. (i) If $x_1 = A \cos \omega_1 t$ and $x_2 = A \cos \omega_2 t$, then

$$x = x_1 + x_2$$

$$= 2A \cos\left(\frac{\omega_2 - \omega_1}{2}\right) t \cos\left(\frac{\omega_2 + \omega_1}{2}\right) t$$

By comparing, $\frac{\omega_2 - \omega_1}{2} = 2.1 \Rightarrow \omega_2 - \omega_1 = 4.2$ and $\frac{\omega_2 + \omega_1}{2} = 50.0 \Rightarrow \omega_2 + \omega_1 = 100.0$

$\Rightarrow 2\omega_2 = 100 + 4.2 \Rightarrow \omega_2 = 52.1$ Hz

(ii) Since, $\omega_2 - \omega_1 = 2\pi(\nu_2 - \nu_1) = 4.2$

Beat frequency, $\nu_2 - \nu_1 = \frac{4.2}{2\pi} = 0.66$ Hz

Ex. 9. At a given point in space the total light wave is composed of three phasors $P_1 = a$, $P_2 = \frac{a}{2}e^{i\theta}$ and $P_3 = \frac{a}{2}e^{-i\theta}$. The intensity of light at this point is

(a) $4a^2\cos^2\left(\frac{\theta}{2}\right)$ (b) $4a^2\cos^4\left(\frac{\theta}{2}\right)$ (c) $a^2\cos^2(\theta)$ (d) $4a\cos^2(2\theta)$

[IIT-JAM : 2006]

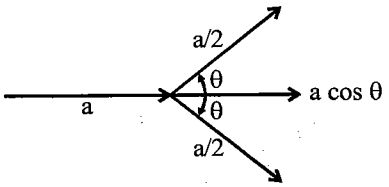
Soln. **(b)** Resultant phasors,

$$P = P_1 + P_2 + P_3 = a + \frac{a}{2}e^{i\theta} + \frac{a}{2}e^{-i\theta}$$

$$P_y = 0 + \frac{a}{2}\sin\theta - \frac{a}{2}\sin\theta = 0$$

$$P_x = a + \frac{a}{2}\cos\theta + \frac{a}{2}\cos\theta$$

$$= a + a\cos\theta = 2a\cos^2\theta/2$$



The intensity of light at this point is $I = P^2 = 4a^2\cos^4\theta/2$

1.4 SUPERPOSITION OF TWO SHMS ACTING AT RIGHT ANGLES TO EACH OTHER:

1.4.1 Oscillations having same frequencies:

Let two simple harmonic motions acting at right angles be represented by the equations

$$x = a\cos(\omega t + \delta_1) \quad \text{and} \quad \ldots (23)$$

$$y = b\cos(\omega t + \delta_2) \quad \ldots (24)$$

$$\Rightarrow \quad \frac{y}{b} = \cos(\omega t + \delta_1 + \delta_2 - \delta_1) = \cos(\omega t + \delta_1 + \delta)$$

where $\delta = \delta_2 - \delta_1$ is the phase difference between the two SHMs.

Therefore, $\frac{y}{b} = \cos(\omega t + \delta_1).\cos\delta - \sin(\omega t + \delta_1).\sin\delta$

$$\Rightarrow \quad \frac{y}{b} = \frac{x}{a}\cos\delta - \sqrt{1 - x^2/a^2}.\sin\delta$$

$$\Rightarrow \quad \left(\frac{x}{a}\cos\delta - \frac{y}{b}\right)^2 = \left(1 - \frac{x^2}{a^2}\right)\sin^2\delta$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\delta = \sin^2\delta \quad \ldots (25)$$

It represents the general equation of an ellipse bounded within a rectangle of sides 2a and 2b. The major axis of the ellipse makes an angle θ with the x-axis, which is given by

$$\tan 2\theta = \frac{2ab}{a^2 - b^2}\cos\delta \quad \ldots (26)$$

Case-I: If $\delta = \delta_2 - \delta_1 = 0$ i.e, the two SHMs are in phase, then the Equation (25) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0 \qquad \Rightarrow \left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

It represents a pair of coincident straight lines $y = \frac{b}{a}x$, passing through the origin and inclined to the x-axis at an angle $\tan^{-1}(b/a)$

Case-II: When $\delta = \pi$, Equation. (25) becomes $\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$

This also represents a pair of coincident straight lines passing through the origin inclined to the x–axis at an angle θ given by $\tan\theta = (-b/a)$.

Case-III: When $\delta = \pi/2$, Equation (25) reduces to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

It represents an ellipse with semi-axes a and b along the coordinate axes. If, in addition, a = b ellipse degenerates into a circle, $x^2 + y^2 = a^2$

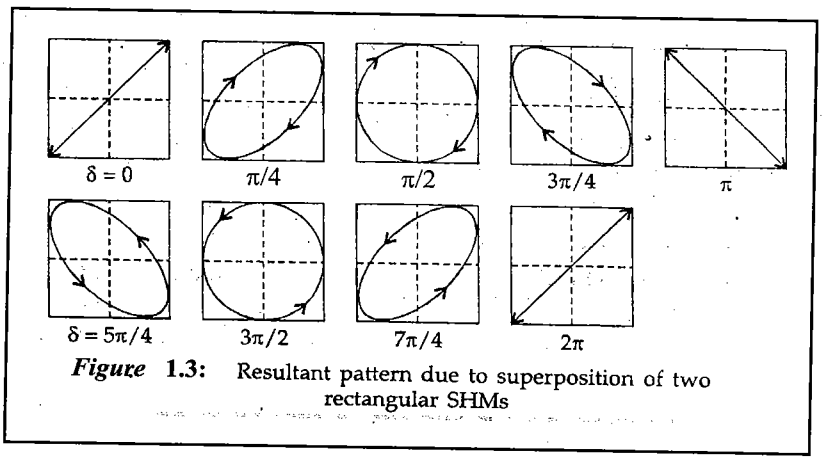


***Figure* 1.3:** Resultant pattern due to superposition of two rectangular SHMs

1.4.2 Oscillations having slightly different frequencies:

Let two rectangular SHMs having slightly different frequencies be represented by

$$x = a\cos(\omega t + \delta_1) \qquad \ldots (27)$$

$$y = b\cos[(\omega + \Delta\omega)t + \delta_2] = b\cos(\omega t + \delta_1 + \delta) \qquad \ldots (28)$$

where $\delta = \Delta\omega t + \delta_2 - \delta_1$ is the phase difference between the two SHMs.

As δ changes from 0 to 2π the resultant pattern goes through all the phases shown in ***Figure*1.3**. As δ goes on changing with time the whole pattern will be repeated in a cyclic fashion. The pattern goes through a cycle of changes in time T given by

$$\delta(t + T) = \delta(t) + 2\pi$$

$$\Rightarrow \quad \Delta\omega T = 2\pi$$

$$\Rightarrow \quad T = \frac{2\pi}{\Delta\omega} = \frac{1}{\Delta\nu} \qquad \ldots (29)$$

1.4.3 Frequency ratio 1 : 2:

Let two rectangular SHMs of frequencies in the ratio 1 : 2 differing in phase by δ be represented by

$$x = a\cos\omega t \qquad \ldots (30)$$

$$y = b\cos(2\omega t + \delta) \qquad \ldots (31)$$

$$\Rightarrow \quad \frac{y}{b} = \cos 2\omega t.\cos\delta - \sin 2\omega t.\sin\delta = (2\cos^2\omega t - 1)\cos\delta - 2\sin\omega t\cos\omega t\cdot\sin\delta$$

$$\Rightarrow \quad \frac{y}{b} = \left(2.\frac{x^2}{a^2} - 1\right)\cos\delta - 2\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\sin\delta$$

$$\Rightarrow \quad \left[\frac{y}{b} + \cos\delta - \frac{2x^2}{a^2}\cos\delta\right]^2 = \frac{4x^2}{a^2}\left(1 - \frac{x^2}{a^2}\right)\sin^2\delta \qquad \ldots (32)$$

This is an equation of fourth degree in x and y, in general, represents closed curve having two loops. For a given value of δ the exact nature of the curve can be traced. For example, if $\delta = 0$ then Equation (32) reduces to

$$\left(\frac{y}{b} + 1 - \frac{2x^2}{a^2}\right)^2 = 0 \qquad \ldots (33)$$

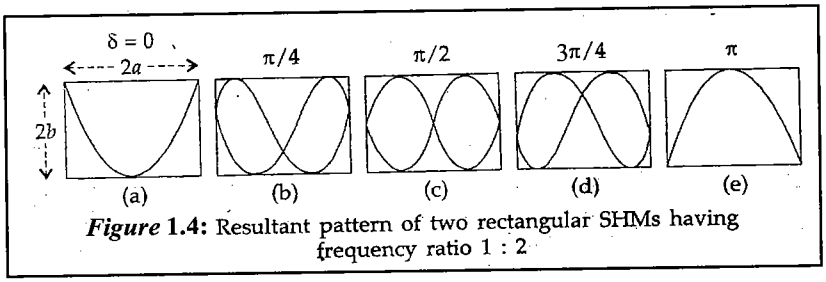


***Figure* 1.4:** Resultant pattern of two rectangular SHMs having frequency ratio 1 : 2

It represents two coincident parabolas (***Figure*** 1.4 (a)) given by

$$x^2 = \frac{a^2}{2b}(y + b) \qquad \ldots (34)$$

If $\delta = \pi/2$ equation (32) reduces to $\frac{4x^2}{a^2}\left(\frac{x^2}{a^2} - 1\right) + \frac{y^2}{b^2} = 0$ ... (35)

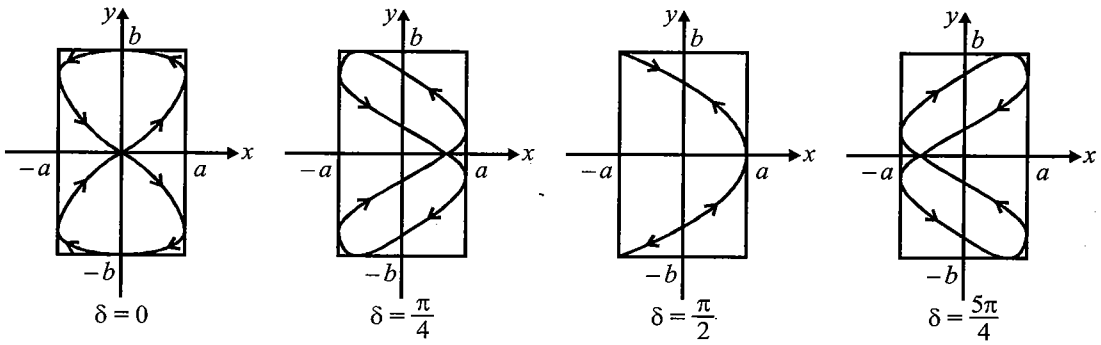
This equation represents a curve containing two loops as shown in ***Figure*** 1.4(c).

Note: If particle displacement are of following form

$x = a\sin(2\omega t + \delta)$, $y = b\sin(\omega t)$, the general equation of resultant curve is given by

$$\left(\frac{x}{a} - \sin\delta\right)^2 + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1 + \frac{x}{a}\sin\delta\right) = 0$$

And resultant Lissajous figures loops are



1.5 LISSAJOUS FIGURES:

The figures or curves formed by the superposition of two simple harmonic motions at right angles to each other are known as Lissajous figures. The shape of these curves depend on the ratio of frequencies as well as on the initial phase relationship of the component simple harmonic motions.

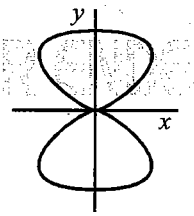
1.5.1 Uses of Lissajous figures:

1. Ratio of frequencies

These figures may be used to find the ratio of two exactly commensurate frequencies of the component vibrations. For this purpose draw a horizontal and a vertical line such that it intersect curve at maximum point and find the number of intersections each line makes with the curve. Now the *ratio of the number of cut of the vertical line to that on the horizontal line will give the ratio of the horizontal to the vertical frequency* i.e.,

$$\frac{\text{Number of cuts on vertical line}}{\text{Number of cuts on horizontal line}} = \frac{N_y}{N_x} = \frac{\omega_x}{\omega_y} \quad \ldots (36)$$

Ex. 10. Superposition of two sinusoidal waves $x(t) = A \sin(\omega_1 t)$ and $y(t) = B \sin(\omega_2 t)$ produces the following Lissajous figure. The ratio ω_1/ω_2 is



(a) 1/2 (b) 1 (c) 2 (d) 4

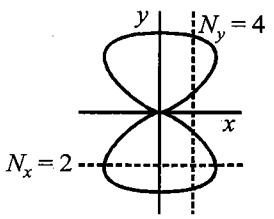
[IIT-JAM GP : 2010]

Soln. (c)

$$\frac{\text{Number of times figure cuts } y\text{-axis}}{\text{Number of times figure cuts } x\text{-axis}} = \frac{N_y}{N_x} = \frac{\omega_x}{\omega_y}$$

$$\Rightarrow \quad \frac{4}{2} = \frac{\omega_x}{\omega_y}$$

$$\omega_x = 2\omega_y$$



So, $\frac{\omega_1}{\omega_2} = 2$

2. Beat frequency

Lissajous figure can be used to compare two nearly equal frequencies. If the frequencies differ slightly Lissajous figures change gradually and pass through a complete cycle of changes in a time T given by

$$T = \frac{1}{\Delta v} \qquad \ldots (37)$$

3. Phase difference

Lissajous figure can also be used to measure the phase difference δ between two signals of same frequency.

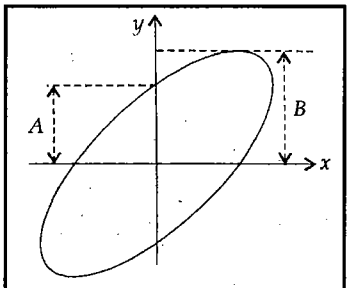


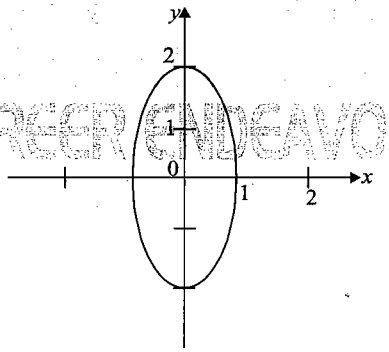
***Figure* 1.5:** Measurement of phase difference

$$\delta = \sin^{-1}(A/B) \qquad \ldots (38)$$

Ex. 11. When two simple harmonic oscillations represented by

$$x = A_0 \cos(\omega t + \alpha) \text{ and } y = B_0 \cos(\omega t + \beta)$$

are superposed at right angle, the result is an ellipse with its major axis along the y-axis as shown in the figure. The conditions which corresponds to this are



(a) $\beta = \alpha + \frac{\pi}{2};\ A_0 = 2B_0$

(b) $\beta = \alpha - \frac{\pi}{4};\ A_0 = B_0$

(c) $\beta = \alpha + \frac{\pi}{2};\ 2A_0 = B_0$

(d) $\beta = \alpha + \frac{\pi}{4};\ A_0 = B_0$

[IIT-JAM : 2007]

Soln. **(c)** The lissajous figure is ellipse only when phase difference is $\frac{\pi}{2}$ and from given figure, amplitude of y-vibration is twice of amplitude of a x-vibration.

$$\left.\begin{aligned}\beta &= \alpha + \frac{\pi}{2} \\ B_0 &= 2A_0\end{aligned}\right]$$

SOLVED EXAMPLES

1. Show that a SHM can be considered as the superposition of two equal and opposite circular motions.

Soln. Circular motions can be represented by

$$\left.\begin{aligned} x_1 &= a\cos\omega t \\ y_1 &= a\sin\omega t \end{aligned}\right\} \text{Anticlockwise} \qquad \left.\begin{aligned} x_2 &= a\cos\omega t \\ y_2 &= -a\sin\omega t \end{aligned}\right\} \text{Clockwise}$$

Adding these equations we get $x = 2a\cos\omega t, y = 0$. It represents a SHM.

2. A Particle of mass m is under a unidimensional potential $V(x) = V_0(1 - \cos Kx)$ where V_0 and K are constants. Find the time period of small oscillations about the equilibrium position.

Soln. For equilibrium, $\frac{dV}{dx} = 0 \quad \Rightarrow \sin Kx = 0 \quad \Rightarrow x = 0$

Now if x be the displacement from equilibrium at any instant then equation of motion is

$$m\frac{d^2x}{dt^2} = -\frac{dV}{dx} = -V_0K\sin Kx$$

For small oscillation $\sin Kx \to Kx$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{V_0K^2}{m}x = -\omega^2x \text{ where } \omega = \sqrt{\frac{V_0k^2}{m}}$$

It represents a SHM. Its time period is $T = \frac{2\pi}{\omega} = \frac{2\pi}{K}\sqrt{\frac{m}{V_0}}$

Alternate solution :

Spring constant $= \left.\frac{d^2V}{dx^2}\right|_{x_0} = V_0K^2, \omega = \sqrt{\frac{V_0K^2}{m}}$

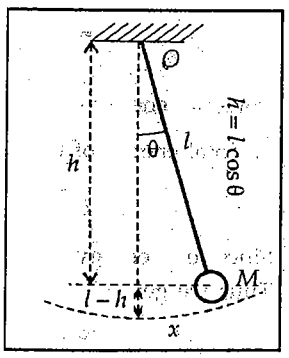
CAREER ENDEAVOUR

3. A simple pendulum consists of a rod of mass m and length l which is capable of oscillation about one end and the other end carries a mass M. Determine the natural frequency of oscillation.

Soln. Let θ be the small angular displacement at any instant as shown in figure. Now the kinetic energy of the system is

$$KE = \frac{1}{2}I_{rod}\dot{\theta}^2 + \frac{1}{2}I_{mass}\dot{\theta}^2$$

$$= \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}^2 + \frac{1}{2}Ml^2\dot{\theta}^2 = \frac{1}{2}\left(M + \frac{m}{3}\right)l^2\dot{\theta}^2$$



Potential energy of the system is

$$PE = Mgl(1 - \cos\theta) + mg\frac{\ell}{2}(1 - \cos\theta)$$

$$= \left(M + \frac{m}{2}\right)gl(1 - \cos\theta)$$

Since total energy (KE + PE) is constant, $\frac{d}{dt}(KE + PE) = 0$

$$\Rightarrow \quad \left(M + \frac{m}{3}\right)l^2\ddot{\theta} + \left(M + \frac{m}{2}\right)gl\sin\theta = 0$$

For small oscillation $\sin\theta \approx \theta$ and the equation of motion becomes,

$$\ddot{\theta} = -\omega^2\theta, \text{ where } \omega^2 = \frac{M + m/2}{M + m/3}.\frac{g}{l}$$

Therefore, the time period of oscillation is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{M + m/3}{M + m/2}.\frac{l}{g}}$

4. A sphere of mass M capable of producing SHM under a restoring force sx is at rest. Another sphere of mass m moving with velocity v_o along x-axis makes a head-on elastic collision with the first sphere. Find the subsequent motion of the first sphere.

Soln. Applying the principles of conservation of linear momentum and energy we can write

$$mv_0 + M.0 = mv + MV \text{ and} \quad \ldots(1)$$

$$\frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv^2 + \frac{1}{2}MV^2 \quad \ldots(2)$$

where v and V are the velocities of m and M respectively along x-axis.

Rearranging Equation (1) and (2) we get,

$$m(v_0 - v) = MV \text{ and} \quad \ldots(3)$$

$$m(v_0^2 - v^2) = MV^2 \quad \ldots(4)$$

Dividing Equation (4) by Equation (3) we get

$$v_0 + v = V \quad \ldots(5)$$

Now from Equation (1) we get $mv_0 = m(V - v_0) + MV$

$$\Rightarrow \quad V = \frac{2m}{m + M}v_0 \quad \ldots(6)$$

CAREER ENDEAVOUR

The equation of motion of M is

$$M\ddot{x} = -sx \quad \Rightarrow \ddot{x} + \omega^2 x = 0 \text{ where } \omega = \sqrt{s/M}$$

The general solution is $x = A\cos\omega t + B\sin\omega t$

To find the constants A and B we apply the initial conditions, $x = 0$ and $dx/dt = V$ at $t = 0$. Thus we get $A = 0$ and $B = V/\omega$.

Therefore, the subsequent motion of M will be described by the equation $x = \frac{V}{\omega}\sin\omega t$

where $\omega = \sqrt{s/M}$ and $V = \frac{2m}{m + M}v_0$.

5. Lissajous figure obtained by using two tuning two forks A and B having slightly different frequencies undergoes complete cycle of changes in 20 sec. When B is loaded with a little wax, the time reduces to 10 sec. If the frequency of A is 288, calculate the frequency of B before loading it with wax.

Soln. The frequency difference before loading is $\Delta v = 1/20$, $v_A \sim v_B = 0.05$ Hz

After loading ν_B decreases to ν_B' and the difference $\nu_A \sim \nu_B' = 1/10 = 0.1$ Hz increases. This indicates that $\nu_A > \nu_B$. Thus, $\nu_A - \nu_B = 0.05\,\text{Hz}$ or, $\nu_B = \nu_A - 0.05\,\text{Hz} = (288 - 0.05)\,\text{Hz} = 287.95\,\text{Hz}$

5. Suppose a particle is simultaneously subjected to three collinear SHMs all of the same frequency and having amplitudes 1 cm, 0.5 cm and 0.25 cm respectively. The phase of the second relative to the first is 30° and that of the third relative to the second is 60°. Find the resultant motion.

Soln. The resultant is

$$x = \cos\omega t + 0.5\cos(\omega t + 30°) + 0.25\cos(\omega t + 90°)$$

$$= \cos\omega t + 0.5\left[\cos\omega t.\frac{\sqrt{3}}{2} - \sin\omega t.\frac{1}{2}\right] - 0.25\sin\omega t = \left(1 + \frac{\sqrt{3}}{4}\right)\cos\omega t - \frac{1}{2}\sin\omega t$$

$$= A\cos(\omega t + \phi)$$

where $A\cos\phi = 1 + \frac{\sqrt{3}}{4} = 1.433$, $A\sin\phi = \frac{1}{2} = 0.5$

Therefore, for the resultant motion, amplitude $A = \sqrt{(1.433)^2 + (0.5)^2} = 1.52$ cm and the phase relative to the first is $\phi = \tan^{-1}\frac{0.5}{1.433} = 19.23°$.

Alternate solution:

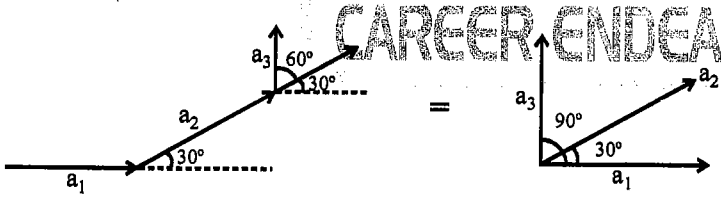
Resultant displacement is given by

$x = a_1\cos\omega t + a_2\cos(\omega t + 30°) + a_3\cos(\omega t + 90°)$,

where $a_1 = 1$ cm, $a_2 = 0.5$ cm, $a_3 = 0.25$ cm.

Taking amplitude as a vector and phase difference as a angle between two vectors resultant amplitude is given by vector sum of a_1, a_2, a_3.

CAREER ENDEAVOUR



From above figure

x-component of amplitude $A_x = a_1 + a_2\cos 30° = 1 + 0.5\frac{\sqrt{3}}{2} = 1.433$

y-component of amplitude $A_y = a_3 + a_2\sin 30° = 0.25 + 0.5\frac{1}{2} = 0.5$

Therefore, resultant amplitude is given by $A = \sqrt{A_x^2 + A_y^2} = \sqrt{(0.5)^2 + (1.433)^2} = 1.52$ cm, and phase with the first SHM is, $\tan\phi = \frac{A_y}{A_x} = \frac{0.5}{1.433} = 0.348$.

$\Rightarrow \quad \phi = \tan^{-1}(0.348) = 19.23°$.

The resultant motion is $x = A\cos(\omega t + \phi)$, where $A = 1.52$.

7. A particle of mass 0.2 kg oscillates according to the law $x = 0.08\cos(20\pi t + \pi/4)$ where x is in metre and t is in second. Find expressions for the instantaneous velocity and acceleration of the particle. What is the instantaneous force acting on it? Determine the amplitudes of velocity, acceleration and force. What is the frequency of oscillation? What is the frequency of variation of its kinetic energy?

Soln. Let $x = A\cos(\omega t + \delta)$

Instantaneous velocity $v = \dfrac{dx}{dt} = -A\omega\sin(\omega t + \delta) = v_0\sin(\omega t + \delta) = -5.02\sin(20\pi t + \pi/4)\,m/s$

where $v_0 = -A\omega = -5.02$ m/s.

Acceleration $a = \dfrac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \delta) = a_0\cos(\omega t + \delta) = -315.5\cos(20\pi t + \pi/4)\,m/s^2.$

Force F = mass × acceleration $= -0.2 \times (20\pi)^2 \times 0.08\cos(20\pi t + \pi/4) = 63.1\cos(20\pi t + \pi/4)\,N$

Therfore, Amplitudes of velocity, acceleration and force are 5.02 m/s, $315.5\,m/s^2$ and 63.1 N respectively.

Frequency of oscillation $\nu = \dfrac{\omega}{2\pi} = \dfrac{20\pi}{2\pi} = 10\,Hz$

Kinetic energy K.E. $= \dfrac{1}{2}m\left(\dfrac{dx}{dt}\right)^2 = \dfrac{1}{2}\ 0.2 \times (5.02)^2\sin^2(20\pi t + \pi/4)\,J$

$= 2.52 \times \dfrac{1}{2}\left[1 - \cos 2(20\pi t + \pi/4)\right]J = 1.26\left[1 - \cos(40\pi t + \pi/2)\right]J$

$= 1.26\left[1 + \sin 40\pi t\right]J$

Therefore, the frequency of variation of kinetic energy is $\dfrac{40\pi}{2\pi} = 20\,Hz$, which is twice the frequency of particle's motion.

CAREER ENDEAVOUR

8. Find the resultant of the following two SHMs:

$x_1 = 0.03\cos 10\pi t,\ x_2 = 0.03\cos 12\pi t$

Hence find the beat frequency and beat amplitude.

Soln. The resultant is given by $x = x_1 + x_2 = 0.03\left[\cos 10\pi t + \cos(10\pi t + 2\pi t)\right]$

$= 0.03(1 + \cos 2\pi t)\cos 10\pi t - 0.03\sin 2\pi t\sin 10\pi t = A\cos(10\pi t + \phi)$

where we have put $0.03(1 + \cos 2\pi t) = A\cos\phi$ and $0.03\sin 2\pi t = A\sin\phi$

Thus, $A = 0.03\sqrt{(1 + \cos 2\pi t)^2 + \sin^2 2\pi t} = 0.03\sqrt{2 + 2\cos 2\pi t}$

Therefore, the required amplitude is $A = 0.03\sqrt{2 + 2 \times 1} = 0.06$ unit.

The beat frequency = difference of the two frequencies

$= \nu_2 - \nu_1 = \dfrac{1}{2\pi}(\omega_2 - \omega_1) = (6 - 5) = 1$ cycle per sec.

9. A particle vibrates harmonically with a frequency 0.5 Hz,. Assuming that the motion is started from its equilibrium position with a speed of 0.2 m/s, towards negative x-axis write down its equation of vibration.

Soln. Let the motion be represented by $x(t) = a\sin(\omega t + \theta_0) = a\sin(2\pi\nu t + \theta_0)$

$= a\sin(\pi t + \theta_0)$ $[\because \nu = 0.5\ \text{Hz}]$

where a and θ_0 are two arbitrary constants to be fixed by the initial conditions. At $t = 0, x = 0$,

$x(t) = 0 = a\sin\theta_0 \Rightarrow \theta_0 = 0, \pi, 2\pi, \ldots.$

Now $\dfrac{dx}{dt} = a\pi\cos(\pi t + \theta_0)$

At $t = 0, \dfrac{dx}{dt} = -0.2\ \text{m/s}$. So, $-0.2 = a\omega\cos\theta_0 \Rightarrow \theta_0 = \pi$

and $a\pi = 0.2 \Rightarrow a = (0.2/\pi)\,\text{m}$

Therefore, the required equation of vibration is $x(t) = (0.2/\pi)\sin(\pi t + \pi)$

10. Two mutually perpendicular oscillations are represented by

$x(t) = a\sin\omega t$ and $y(t) = b\sin(2\omega t + \phi)$

Sketch the Lissajous figure resulting from these oscillations with $\phi = -\pi$. Find the value of x for which the Lissajous figure touches the line $y = \pm b$.

Soln. $x(t) = a\sin\omega t, y(t) = b\sin(2\omega t - \pi) = -b\sin 2\omega t = -2b\sin\omega t.\cos\omega t$

Using, $\sin\omega t = x/a, \cos\omega t = \sqrt{1 - x^2/a^2}$ we get,

$$y = -2b.\frac{x}{a}.\sqrt{1 - x^2/a^2}$$

$$\Rightarrow \quad \frac{y^2}{b^2} = \frac{4x^2}{a^2}\left(1 - \frac{x^2}{a^2}\right)$$

This equation represents a curve containing two loops as shown in ***Figure*** 1.4(c).

Putting $y = \pm b$ we get

$$\frac{4x^4}{a^4} - \frac{4x^2}{a^2} + 1 = 0 \quad \Rightarrow \quad \left(\frac{2x^2}{a^2} - 1\right)^2 = 0 \quad \Rightarrow \quad x = \pm\frac{a}{\sqrt{2}}$$

These are the required values of x.

11. A mass M is attached to a solid support with an ideal spring constant k. The natural frequency of the system is ω. If the mass now carries a charge $+Q$ and external electric field E along the axis of the spring is turned on, the frequency for small oscillations

(a) becomes zero (b) increases (c) decreases (d) remains unchanged

Ans. (d) This is because the force due to the electric field is a constant one and does not depend on the displacement of the mass from the mean position. Hence, the force due to the electric field will not be able to change the frequency of the SHM. Restoring force, which is present here is only spring force. Therefore, equation of motion of SHM do not changes. Thus frequency remain constant.

PRACTICE SET

[Multiple Choice Questions (MCQ)]

1. A uniform spring whose unstressed length is l, has a constant K. The spring is cut into two pieces of unstressed length l_1 and l_2, where $l_2 = nl_1$, n being an integer. Now a mass m is made to oscillate with first spring. The time period of its oscillation would be

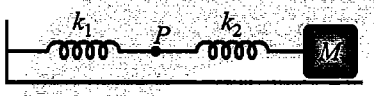
(a) $T = 2\pi\sqrt{\frac{mn}{K(n+1)}}$ (b) $T = 2\pi\sqrt{\frac{m}{nK}}$

(c) $T = 2\pi\sqrt{\frac{m}{K(n+1)}}$ (d) $T = 2\pi\sqrt{\frac{m(n+1)}{nK}}$

2. If the displacement (x) and velocity v of a particle executing simple harmonic motion are related through the expression $4v^2 = 25 - x^2$, then its time period is

(a) π (b) 2π (c) 4π (d) 6π

3. The mass M shown in the figure oscillates in simple harmonic motion with amplitude A. The amplitude of the point P is



(a) $\frac{k_1 A}{k_2}$ (b) $\frac{k_2 A}{k_1}$ (c) $\frac{k_1 A}{k_1 + k_2}$ (d) $\frac{k_2 A}{k_1 + k_2}$

4. A point mass is subjected to two simultaneous sinusoidal displacements in x-direction, $x_1(t) = A\sin\omega t$ and $x_2(t) = A\sin\left(\omega t + \frac{2\pi}{3}\right)$. Adding a third sinusoidal displacement $x_3(t) = B\sin(\omega t + \phi)$ brings the mass to a complete rest. The values of B and ϕ are

(a) $\sqrt{2}A, \frac{3\pi}{4}$ (b) $A, \frac{4\pi}{3}$ (c) $\sqrt{3}A, \frac{5\pi}{6}$ (d) $A, \frac{\pi}{3}$

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5. A particle of mass m is executing oscillation about the origin on the x-axis. Its potential energy is $U(x) = k|x|^3$, where k is a positive constant. If the amplitude of oscillation is a, then its time period T is

(a) proportional to $\frac{1}{\sqrt{a}}$ (b) independent of a

(c) proportional to $\sqrt{a}$ (d) proportional to $a^{3/2}$

6. The period of oscillation of simple pendulum of length L suspended from the roof of the vehicle which moves without friction, down an inclined plane of inclination α, is given by:

(a) $2\pi\sqrt{\frac{L}{g\cos\alpha}}$ (b) $2\pi\sqrt{\frac{L}{g\sin\alpha}}$ (c) $2\pi\sqrt{\frac{L}{g}}$ (d) $2\pi\sqrt{\frac{L}{g\tan\alpha}}$

7. The function $x = A\sin^2\omega t + B\cos^2\omega t + C\sin\omega t\cos\omega t$ represents SHM

(a) for every value of A, B and C (except $C = 0$)

(b) if $A = -B$, $C = B$, amplitude $= |B\sqrt{2}|$

(c) if $A = B, C = 0$

(d) if $A = B, C = 3B$, amplitude $= |B|$

. A particle oscillates simple harmonically along x-axis. At times t, $2t$ and $3t$ it is located at a distance a, b and c respectively from the equilibrium position. The period of oscillation is

(a) $T = \dfrac{2\pi t}{\cos^{-1}\left(\dfrac{a+c}{2b}\right)}$ (b) $T = \dfrac{2\pi t}{\cos^{-1}\left(\dfrac{ac}{2b}\right)}$ (c) $T = \dfrac{2\pi t}{\cos^{-1}\dfrac{a}{2b+c}}$ (d) $T = \dfrac{\pi t}{\cos^{-1}\left(\dfrac{a+c}{2b}\right)}$

. A particle of mass m is located in a potential $u(x) = \dfrac{2}{x^2} - \dfrac{4}{x}$. The period of oscillation is

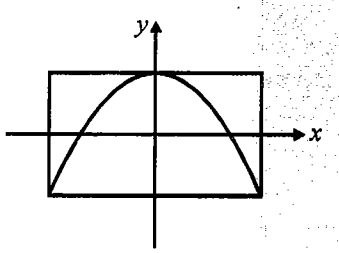
(a) $T = \dfrac{\pi}{4}\sqrt{4m}$ (b) $T = \dfrac{\pi}{2}\sqrt{2m}$ (c) $T = \dfrac{\pi}{2}\sqrt{4m}$ (d) $T = \dfrac{\pi}{2}\sqrt{m}$

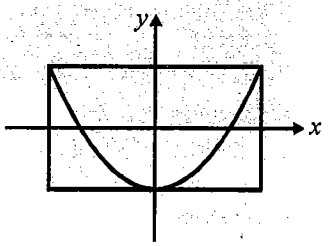
0. A system with potential energy $V(x)$ will perform SHM if

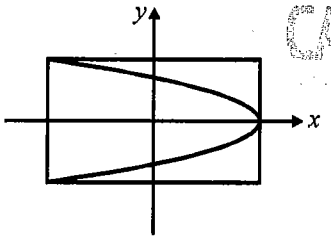
(a) $\dfrac{\partial v}{\partial x} = 0, \dfrac{\partial^2 v}{\partial x^2} > 0$ (b) $\dfrac{\partial v}{\partial x} = 0, \dfrac{\partial^2 v}{\partial x^2} < 0$ (c) only $\dfrac{\partial v}{\partial x} = 0$ (d) only $\dfrac{\partial^2 v}{\partial x^2} < 0$

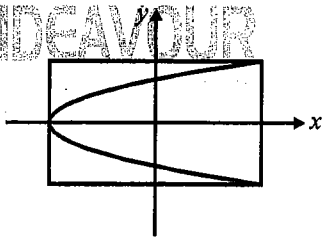
1. A particle is subjected to two SHM's represented by the following equations

$x = a\sin(\omega t)$, $y = b\sin(2\omega t + \delta)$. The Lissajous figure for $\delta = \dfrac{\pi}{2}$ is

(a) 

(b) 

(c) 

(d) 

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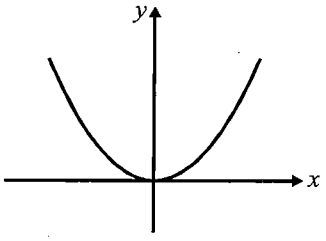
2. A simple pendulum consisting of a ball of mass m tied to a string of length l is made to swing on a circular arc of angle θ in a vertical plane. At the end of this arc, another ball of mass m is placed at rest. The momentum translated to this ball at rest by the swinging ball is

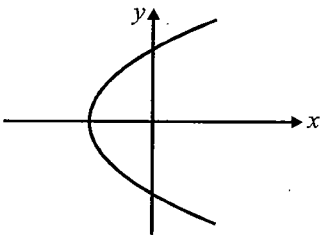
(a) zero (b) $m\theta\sqrt{l/g}$ (c) $m\theta\sqrt{2l/g}$ (d) $\dfrac{m}{2}\sqrt{(l/g)}$

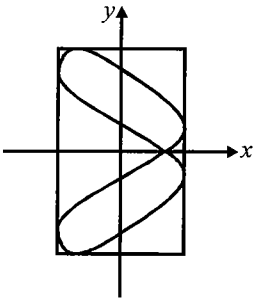
3. A particle of mass m moves in two dimensions under the following potential energy function:

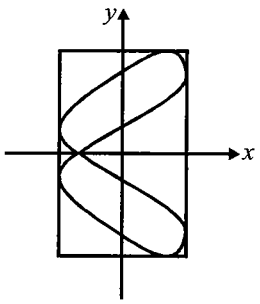
$V(\vec{r}) = \dfrac{1}{2}k(x^2 + 4y^2)$

The possible Lissajous figure formed is

(a) 

(b) 

(c) 

(d) 

14. A simple pendulum with a solid metal has a period T. The metal bob is now immersed in a liquid with a density one-tenth that of the metal of the bob. The liquid is non-viscous. Now the period of the same pendulum with its bob remaining all the time in the liquid will be :

(a) $\left(\frac{9}{10}\right)T$ (b) $T\sqrt{\left(\frac{10}{9}\right)}$ (c) unchanged (d) $T\sqrt{\left(\frac{9}{10}\right)}$

15. The coordinates of a particle moving in a plane are given by $x(t) = a\cos pt$, $y(t) = b\sin pt$ where $a, b (< a)$ and p are positive constants of appropriate dimensions. Then

(a) the path of the particle is a straight line

(b) the velocity and acceleration of the particle are normal to each other at $t = \frac{\pi}{(2p)}$

(c) the acceleration of the particle is always directed towards a focus

(d) the distance travelled by the particle in time interval $t = 0$ to $t = \frac{\pi}{2p}$ is a

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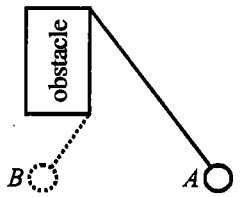
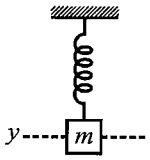
[Numerical Answer Type (NAT)]

1. A simple pendulum has a time period $T = 2$ sec when it swings freely. The pendulum is hung as shown in figure, so that only one-fourth of its total length is free to swing to the left of obstacle. It is displaced to position A and released. How long does it take to swing to extreme displacement B and return to A ?
Assume that displacement angle is always small.



2. An object of mass 0.2 kg executes simple harmonic oscillational along the x-axis with a frequency of $(25/\pi)$ Hz. At the position $x = 0.04$, the object has kinetic energy of 0.5 J and potential energy 0.4 J. The amplitude of oscillations is_____m.

3. A mass m is undergoing SHM in the vertical direction about the mean position y_0 with amplitude A and angular frequency $\omega = 1$ rad/sec. At a distance y from the mean position, the mass detaches from the spring. Assume that the spring contracts and does not obstruct the motion of m.



The distance y_____(measured from the mean position) such that the height h attained by the block is maximum. $(A\omega^2 > g)$.

4. The frequency of the combined motion of the following displacement $\sin\left(2\pi t - \frac{\pi}{4}\right) + \cos(2\pi t)$ is_____Hz.

5. In an experiment to obtain Lissajous' figures, one tuning fork of higher frequency 250 Hz and a circular figure occurs after five seconds. The frequency of the other tuning fork is_____.

ANSWER KEY

[Multiple Choice Questions (MCQ)]

1. (c)	**2. (c)**	**3. (d)**	**4. (b)**	**5. (a)**
6. (a)	**7. (a)**	**8. (a)**	**9. (c)**	**10. (a)**
11. (a)	**12. (a)**	**13. (a)**	**14. (b)**	**15. (b)**

[Numerical Answer Type (NAT)]

1. (1.5 sec)	**2. (0.06)**	**3. (9.8)**	**4. (1 Hz)**

5. (249.8 to 250.2 Hz)

CAREER ENDEAVOUR

Chapter 2

Damped And Force Vibrations

2.1 DAMPED SIMPLE HARMONIC MOTION:

Let us consider the motion of a simple harmonic oscillator in the presence of damping force. The damping force arises due to frictional, viscous and other effects and acts in opposition to the motion. In the simplest situation the damping force may be taken to be proportional to the instantaneous velocity.

2.1.1 Mathematical Analysis of Damped SHM:

Suppose a particle of mass m is oscillating along x-axis under the action of a restoring force proportional to the displacement (x) from the position of equilibrium and a damping force proportional to the instantaneous velocity (dx/dt). The equation of motion of such an oscillator is

$$m\frac{d^2x}{dt^2} = -sx - R_m\frac{dx}{dt} \qquad \ldots (1)$$

where s is the restoring force per unit displacement and R_m is the damping force per unit velocity.

$$\Rightarrow \qquad \frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \qquad \ldots (2)$$

where $2b = R_m / m$ and $\omega_0 = \sqrt{s/m}$. ω_0 may be identified as the angular frequency of vibration in the absence of damping.

To solve the equation let $x = ce^{\alpha t}$ be a trial solution, where c and α are arbitrary constants. Differentiating we get

CAREER ENDEAVOUR

$$\frac{dx}{dt} = \alpha . ce^{\alpha t} \text{ and } \frac{d^2x}{dt^2} = \alpha^2 . ce^{\alpha t}$$

$$\Rightarrow \qquad \left(\alpha^2 + 2b\alpha + \omega_0^2\right)ce^{\alpha t} = 0 \qquad \Rightarrow \alpha^2 + 2b\alpha + \omega_0^2 = 0$$

It has two roots,

$$\alpha_1 = -b + \sqrt{b^2 - \omega_0^2} \text{ and } \alpha_2 = -b - \sqrt{b^2 - \omega_0^2}$$

$$x_1 = c_1 e^{\alpha_1 t} \text{ and } x_2 = c_2 e^{\alpha_2 t},$$

So, its general solution is given by the principle of superposition as

$$x = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t} = e^{-bt}\left[c_1 e^{\sqrt{b^2 - \omega_0^2}\cdot t} + c_2 e^{-\sqrt{b^2 - \omega_0^2}\cdot t}\right] \qquad \ldots (3)$$

Large damping : $b > \omega_0$

In this case $b^2 - \omega_0^2$ is positive quantity and hence the solution is *non-oscillatory*. However, the actual behaviour depends upon the initial conditions.

Initial condition:

Case-I : At $t = 0$,

Position $x = x_0$ and, Velocity $dx/dt = v_0$.

Putting these conditions in Equation (3) we get

$$x_0 = c_1 + c_2$$

$$v_0 = \alpha_1 c_1 + \alpha_2 c_2 \qquad \ldots (4)$$

Solving these equation we get

$$c_1 = \frac{\alpha_2 x_0 - v_0}{\alpha_2 - \alpha_1} = \frac{x_0}{2}\left(1 + \frac{b + v_0/x_0}{\sqrt{b^2 - \omega_0^2}}\right)$$

and $$c_2 = \frac{v_0 - \alpha_1 x_0}{\alpha_2 - \alpha_1} = \frac{x_0}{2}\left(1 - \frac{b + v_0/x_0}{\sqrt{b^2 - \omega_0^2}}\right)$$

Thus from Equation (3) we get

$$x = \frac{x_0}{2} e^{-bt}\left[\left(1 + \frac{b + v_0/x_0}{\sqrt{b^2 - \omega_0^2}}\right) e^{\sqrt{b^2 - \omega_0^2}\,t} + \left(1 - \frac{b + v_0/x_0}{\sqrt{b^2 - \omega_0^2}}\right) e^{-\sqrt{b^2 - \omega_0^2}\,t}\right] \qquad \ldots (5)$$

As $b^2 - \omega_0^2$ is a positive quantity the motion is obviously non-oscillatory, aperiodic or dead beat. If the motion is started with an initial displacement but no initial velocity the displacement gradually falls off to zero with time and the body returns to the equilibrium position without any oscillation about the equilibrium position.

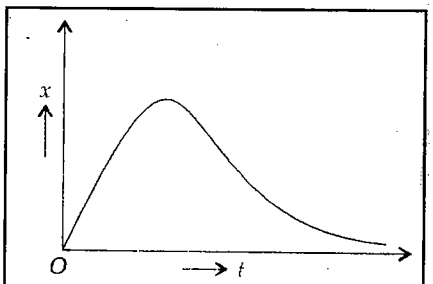


Figure 2.1: Non-oscillatory behaviour of a heavily damped oscillator

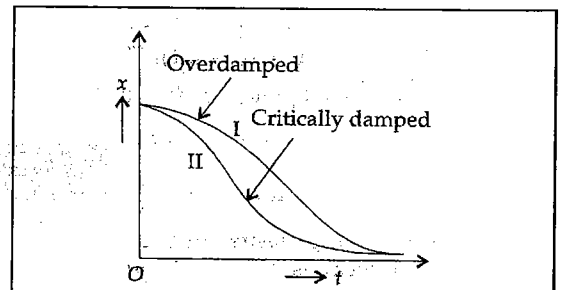


Figure 2.2: Non-oscillatory behaviour of overdamped and critically damped oscillator('t' measured from the instant of maximum displacement)

Case-II : At $t = 0, x = 0$ and $dx/dt = v_0$

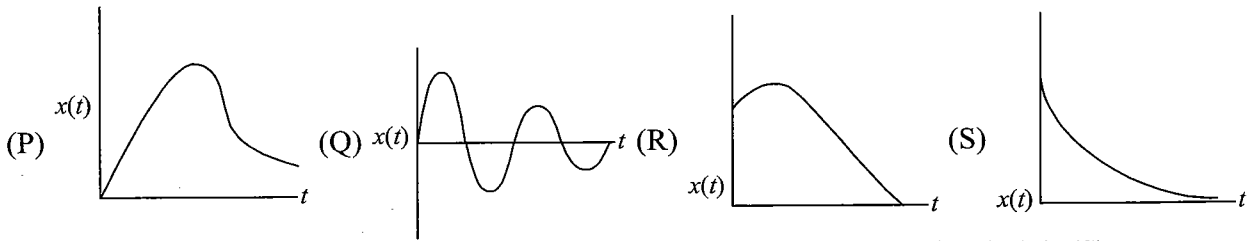
$$c_1 = -c_2 = -\frac{v_0}{\alpha_2 - \alpha_1} = \frac{v_0}{2\sqrt{b^2 - \omega_0^2}}$$

Then $$x(t) = \frac{v_0}{2 \cdot \sqrt{b^2 - \omega_0^2}} e^{-bt}\left[e^{\sqrt{b^2 - \omega_0^2}\,t} - e^{-\sqrt{b^2 - \omega_0^2}\,t}\right]$$

$$= \frac{v_0}{\sqrt{b^2 - \omega_0^2}} e^{-bt} \sinh\sqrt{b^2 - \omega_0^2}\,t \qquad \ldots (6)$$

Describes the behaviour of a heavily damped system which is disturbed from equilibrium by a sudden impulse at t = 0. The motion is illustrated in figure (2.2). For small 't' the term e^{-bt} very nearly equal to unity. Displacement increases with time as $\sinh\sqrt{b^2-\omega_0^2}\,t$ increases with 't'. The displacement reaches a maximum value and thereafter the term e^{-bt} dominates and the displacement decays exponentially with time. This type of motion is found in a dead beat galvanometer or a pendulum immersed in a highly viscous liquid.

Ex. 1. Among the following displacement versus time plots, which ones may represent an overdamped oscillator?



(a) Only (P) and (Q) (b) Only (P) and (R) (c) Only (P) and (S) (d) Only (P), (R), (S)

[IIT-JAM : 2009]

Soln. **(d)** Over damped oscillator does not have oscillatory motion, Therefore any of the (P),(R) and (S) can represent its motion.

Critical damping $b = \omega_0$:

It is a limiting case of heavy damping when the motion changes from non-oscillatory to oscillatory nature. Putting $b = \omega_0$ in equation (2) we find that $\alpha_1 = \alpha_2 = -\omega_0$. As the two roots are equal to the general solution of equation (2) in this case would be given by

$$x = (c_1 + c_2 t)e^{-\omega_0 t} \qquad \ldots (7)$$

Initial condition:

Case-I : At $t = 0$, $x = x_0$ and $dx/dt = v_0$,

then $x_0 = c_1$ and

$$v_0 = -\omega_0 c_1 + c_2 \quad \Rightarrow c_2 = v_0 + \omega_0 x_0$$

Thus, $$\mathbf{x = e^{-\omega_0 t}\left[x_0 + (v_0 + \omega_0 x_0)t\right]} \qquad \ldots (8)$$

The motion represented by the equation (8) is also non-oscillatory. But here the rate of decay is much faster than the overdamped case. The motion is now said to be critically damped.

Case-II : At $t = 0$, $x = x_0$ and $dx/dt = 0$ then

$$\mathbf{x = x_0(1+\omega_0 t)e^{-\omega_0 t},\ v = x_0\omega^2 t\, e^{-\omega_0 t}} \qquad \ldots (9)$$

Maximum velocity at $t = 1/\omega_0$

The displacement decreases from x_0 with time. This is illustrated by the curve II in figure (2.2). In fact in this case particle returns most quickly to the position of equilibrium.

Case-III : At $t = 0$, $x = 0$ and $dx/dt = v_0$ then we have

$$\mathbf{x = v_0 t.e^{-\omega_0 t}}$$

And $$\frac{dx}{dt} = v_0 e^{-\omega_0 t}(1-\omega_0 t) \qquad \ldots (10)$$

In this case initially the displacement increases almost linearly with time and becomes maximum i.e., $dx/dt = 0$ at $t = 1/\omega_0$. Therefore, the displacement decays exponentially with time and ultimately becomes zero (figure (2.3)). Note that the time $t = 1/\omega_0$ is independent of initial velocity. The maximum displacement is

$$x_{max} = v_0 \cdot \frac{1}{\omega_0} e^{-\omega_0 \cdot \frac{1}{\omega_0}} = \frac{v_0}{e.\omega_0} \qquad \ldots (11)$$

The critically damped condition is useful if we desire most quick decay without oscillation. In a pointer-type galvanometer we want the pointer to move smoothly and quickly to a new equilibrium position when current is sent through it. This helps us to take the reading immediately after the meter is connected to a circuit.

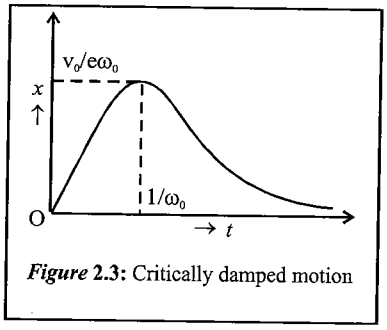


***Figure* 2.3:** Critically damped motion

Q. 2. A spring - mass system has undamped natural angular frequency $\omega_0 = 100$ rad s^{-1}. The solution x(t) at critial damping is given by $x(t) = x_0(1+\omega_0 t)\exp(-\omega_0 t)$, where x_0 is a constant. The system experiences the maximum damping force at time

(a) 0.01 s (b) 0.1 s (c) 0.01 πs (d) 0.1 πs

[IIT-JAM : 2006]

Sln. **(a)** Damping force is maximum when speed is maximum.

i.e. $\frac{d}{dt}\left(\frac{dx}{dt}\right) = 0 \quad \therefore -x_0 w_0^3 \exp(-w_0 t) + w_0^3 x t \exp(-w_0 t) = 0$

$\therefore \quad t = \frac{1}{w_0}; \frac{1}{100} = 0.01\,\text{sec}.$

Alternate Solution:

Let displacement be

$x = x_0(1+\omega_0 t)\exp(-\omega_0 t)$

Damping force $F = -p\frac{dx}{dt} = -p\left[x_0(0+\omega_0)\exp(-\omega_0 t) + x_0(1+\omega_0 t)(-\omega_0)\exp(-\omega_0 t)\right]$

$= -px_0\omega_0\left[\exp(-\omega_0 t) - \exp(-\omega_0 t) - \omega_0 t\exp(-\omega_0 t)\right]$

$= px_0\omega_0^2 t\exp(-\omega_0 t)$

Let F is maximum when $t = t_0$, then $\frac{dF}{dt\,|\,t_0} = 0$

$px_0\omega_0^2\left[\exp(-\omega_0 t_0) - \omega_0 t\exp(-\omega_0 t_0)\right] = 0$

$\Rightarrow \omega_0 t_0 = 1 \Rightarrow t_0 = \frac{1}{\omega_0} = 0.01\,\text{sec}.$

Small damping $b < \omega_0$:

The most important kind of behaviour in acoustics is obtained when $b < \omega_0$. In this case $b^2 - \omega_0^2$ is negative and we an write $\sqrt{b^2 - \omega_0^2} = \sqrt{-\left(\omega_0^2 - b^2\right)} = j\omega$ where $\omega = \sqrt{\omega_0^2 - b^2}$.

The solution (3) now takes the form

$$x = e^{-bt}\left[c_1 e^{j\omega t} + c_2 e^{-j\omega t}\right] = e^{-bt}\left[B_1 \cos \omega t + B_2 \sin \omega t\right]$$

where $B_1 = c_1 + c_2$ and $B_2 = j(c_1 - c_2)$

Putting $B_1 = A\cos\phi$ and $B_2 = A\sin\phi$, we can write

$$x = Ae^{-bt}\cos(\omega t - \phi) \qquad \ldots (12)$$

where A and ϕ are real constants to be evaluated from the given initial conditions. Equation (12) represents a damped oscillatory motion with an angular frequency $\omega = \sqrt{\omega_0^2 - b^2}$ whose amplitude, Ae^{-bt}, diminishes exponentially with time. Obviously the presence of damping reduces the frequency slightly and causes the amplitude to decay exponentially with time.

To evaluate the constants A and ϕ we are to apply initial conditions. Let at $t = 0$, $x = x_0$ and $dx/dt = v_0$. Therefore, from equation (12)

$$x_0 = A\cos\phi \qquad \Rightarrow \; v_0 = -Ab\cos\phi + A\omega\sin\phi$$

$$\Rightarrow \qquad \frac{v_0 + bx_0}{\omega} = A\sin\phi$$

Combining these two equations, we get

$$A = \sqrt{x_0^2 + \left(v_0 + bx_0\right)^2/\omega^2}$$

and $$\tan\phi = \frac{v_0 + bx_0}{\omega x_0} \qquad \ldots(13)$$

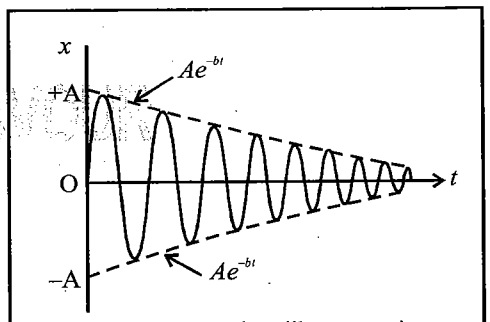


***Figure* 2.4:** Damped oscillatory motion

Case-I: At $t = 0$, $x = x_0$ and $dx/dt = v_0 = 0$, then

$$A = x_0\sqrt{1 + \frac{b^2}{\omega^2}} = x_0\sqrt{1 + \frac{b^2}{\omega_0^2 - b^2}} = \frac{x_0\omega_0}{\sqrt{\omega_0^2 - b^2}}$$

and $$\tan\phi = \frac{b}{\omega} = \frac{b}{\sqrt{\omega_0^2 - b^2}} \qquad \ldots (14)$$

Therefore, from equation (12)

$$x = \frac{x_0\omega_0}{\sqrt{\omega_0^2 - b^2}} e^{-bt}\cos\left(\sqrt{\omega_0^2 - b^2}\cdot t - \tan^{-1}\frac{b}{\sqrt{\omega_0^2 - b^2}}\right) \ldots (15)$$

Case-II: At $t = 0$, $x = x_0 = 0$ and $dx/dt = v_0$, then $A = \frac{v_0}{\omega} = \frac{v_0}{\sqrt{\omega_0^2 - b^2}}$ and $\phi = 90^\circ$

In this case, which is common, the motion is given by equation (12) as

$$x = \frac{v_0}{\sqrt{\omega_0^2 - b^2}} e^{-bt} \sin\sqrt{\omega_0^2 - b^2}\, t \qquad \ldots (16)$$

To discuss the motion of a weakly damped oscillator we plot the solution (16) as shown in figure (2.4). Obviously the motion is oscillatory whose amplitude decays exponentially with time.

Ex. 3. The instantaneous position $x(t)$ of a small block performing one-dimensional damped oscillation is $x(t) = Ae^{-\gamma t}\cos(\omega t + \alpha)$. Here, ω is the angular frequency, γ the damping coefficient, A the initial amplitude and α the initial phase. If $x|_{t=0} = 0$ and $\left.\frac{dx}{dt}\right|_{t=0} = -v$, the values of A and α (with $n = 0, 1, 2, \ldots\ldots$) are

(a) $A = \frac{v}{2\omega}, \alpha = \frac{(2n+1)\pi}{2}$ (b) $A = \frac{v}{\omega}, \alpha = n\pi$

(c) $A = \frac{v}{\omega}, \alpha = \frac{(2n+1)\pi}{2}$ (d) $A = \frac{2v}{\omega}, \alpha = \frac{(2n+1)\pi}{2}$

[IIT-JAM : 2008]

Soln. **(c)** $x|_{t=0} = 0 \Rightarrow A\cos(\alpha) = 0 \Rightarrow \boxed{\alpha = (2n+1)\frac{\pi}{2}}$;

$$\Rightarrow \frac{dx}{dt} = -A\, e^{-\gamma t}\omega\sin(\omega t + \alpha) - \gamma A e^{-\gamma t}\cos(\omega t + \alpha)$$

$$\left.\frac{dx}{dt}\right|_{t=0} = v = A\omega \Rightarrow \boxed{A = \frac{v}{\omega}}$$

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2.1.2 Amplitude decay constant (τ):

It is defined as the time in which the amplitude decays to 1/e of its initial value. As the amplitude at any instant is Ae^{-bt}, the amplitude decay constant τ is given by

$$\frac{Ae^{-b(t+\tau)}}{Ae^{-bt}} = \frac{1}{e} \Rightarrow \tau = \frac{1}{b} \qquad \ldots (17)$$

τ is also called **relaxation time** or modulus of decay.

2.1.3 Logarithmic decrement:

Another method of measuring damping effect on the motion is in terms of logarithmic decrement. Let at time $t = t_1$ amplitude $A_1 = Ae^{-bt_1}$. If $T = 2\pi/\omega$ is the time period of oscillation, then half time period later the amplitude A_2 is given by

$$A_2 = Ae^{-b\left(t_1+\frac{T}{2}\right)} = A_1 e^{-\frac{bT}{2}}$$

Thus $\frac{A_1}{A_2} = e^{bT/2} = d(\text{say})$ or, $\log_e(d) = \frac{bT}{2} = \lambda(\text{say})$.

The quantity d which is the ratio of two successive amplitudes (on opposite sides of the mean position) is called the decrement and $\log_e d = \lambda$ *is called logarithmic decrement of the damped oscillation.* The logarithmic decrement may be defined as the natural logarithm of the ratio of two successive amplitudes of damped oscillations that are separated by half the time period. If A_1 and A_{n+1} are the amplitudes for the first and $(n)^{th}$ half oscillation then

$$\frac{A_1}{A_{n+1}} = \frac{A_1}{A_2}.\frac{A_2}{A_3}.....\frac{A_n}{A_{n+1}} = (d)^n$$

$$\Rightarrow \quad \boldsymbol{\lambda = \log_e d = \frac{1}{n}\log_e \frac{A_1}{A_{n+1}}} \qquad \text{... (18)}$$

Note : Some Text book define logarithmic decrement as, $\lambda = \frac{2}{n}\log\frac{A_1}{A_{n+1}} = bT$

2.1.4 Energy of a damped simple harmonic oscillator:

The P.E. of the particle for a displacement x is

$$E_p = \int_0^x s \cdot x\,dx = \frac{1}{2}sx^2 \qquad \text{... (19)}$$

The K.E. at that instant is

$$E_k = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 \qquad \text{... (20)}$$

Therefore, total energy $E = E_k + E_p$

$$= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}sx^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}m\omega_0^2 x^2 \qquad \text{... (21)}$$

Now from equation (12) $x = Ae^{-bt}\cos(\omega t - \phi)$

$$\Rightarrow \quad \frac{dx}{dt} = -Abe^{-bt}\cos(\omega t - \phi) - A\omega e^{-bt}\sin(\omega t - \phi)$$

$$\Rightarrow E = \frac{1}{2}mA^2e^{-2bt}\left[b^2\cos^2(\omega t - \phi) + \omega^2\sin^2(\omega t - \phi) + b\omega\sin 2(\omega t - \phi) + \omega_0^2\cos^2(\omega t - \phi)\right]$$

... (22)

Therefore, time averaged energy of the oscillator is

$$< E > = \frac{1}{2}mA^2e^{-2bt}\left[b^2.\frac{1}{2} + \omega^2.\frac{1}{2} + 0 + \omega_0^2.\frac{1}{2}\right]$$

$$= \frac{1}{2}mA^2\omega_0^2e^{-2bt} = E_0e^{-2bt} \quad \left[\because \omega = \sqrt{\omega_0^2 - b^2}\right] \qquad \text{... (23)}$$

where $E_0 = \frac{1}{2}mA^2\omega_0^2$ is the total energy of an undamped oscillator. So the total energy of a weakly damped oscillator decays exponentially with time.

Energy decay constant :

The initial energy falls to 1/e of its initial value after a time $\tau' = 1/2b$, known as energy decay time constant.

2.1.5 Equation of motion from energy consideration:

Total instantaneous energy

$$E = K.E. + P.E. = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}sx^2 \qquad ...(24)$$

In the presence of damping E does not remain constant because a part of it used to overcome the damping forces. The rate of loss of energy must be equal to the rate of work done against damping forces. Thus

$$-\frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}sx^2\right] = R_m\frac{dx}{dt}.\frac{dx}{dt}$$

$$\Rightarrow \quad -\frac{dx}{dt}\left[m\frac{d^2x}{dt^2} + sx\right] = R_m\frac{dx}{dt}.\frac{dx}{dt} \qquad \Rightarrow \frac{dx}{dt}\left[m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + sx\right] = 0$$

Since dx/dt is not zero for all values of t,

$$m\frac{d^2x}{dt^2} + R_m\frac{dx}{dt} + sx = 0$$

which is the differential equation of motion of a damped harmonic oscillator.

The rate of loss of average energy:

$$-\frac{d<E>}{dt} = 2bE_0e^{-2bt} = 2b<E> \qquad ...(25)$$

The energy loss is due to work done against damping force. So this must be equal to the average rate of work done against damping force. The rate of work done against damping force is

$$R_m\frac{dx}{dt}.\frac{dx}{dt} = R_m\left(\frac{dx}{dt}\right)^2$$

$$= R_m.Ae^{-2bt}\left[b^2\cos^2(\omega t - \phi) + \omega^2\sin^2(\omega t - \phi) + b\omega\sin 2(\omega t - \phi)\right] \qquad ...(26)$$

As before, the average rate of work done against damping force is

$$<R_m\left(\frac{dx}{dt}\right)^2> = R_mA^2e^{-2bt}\left[b^2\frac{1}{2} + \omega^2\frac{1}{2} + 0\right] = 2bm.Ae^{-2bt}\frac{1}{2}\left(b^2 + \omega_0^2 - b^2\right)$$

$$= 2b\frac{1}{2}mA^2\omega_0^2e^{-2bt} = 2b<E> \qquad ...(27)$$

Thus we prove that the rate of loss of energy is equal to the rate of work done against damping force.

2.1.6 Quality factor or Q-value of the damped oscillator:

The effect of damping on the motion of the oscillator can be expressed in terms of a quality factor defined as

$$Q = 2\pi \times \frac{\text{Average energy stored in a system}}{\text{Average energy lost per cycle}} = 2\pi \times \frac{<E>}{2b<E>.T}$$

$$= \frac{\omega}{2b} = \frac{\omega m}{R_m} = \omega\tau' \qquad ...(28)$$

Thus smaller is the damping in the system, higher is the value of the quality factor and the energy decay time constant τ'.

Ex. 4. The motion of a one-dimensional damped oscillator is described by the differential equation $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$, where ω_0 is the natural angular frequency of the oscillator and β is the damping parameter. The oscillator is given an initial velocity V_0 at time $t = 0$.

(a) Find an expression for the displacement $x(t)$ of the oscillator.

(b) If $V_0 = 0.13$ m/sec, $\omega_0 = 13\ \text{sec}^{-1}$ and $\beta = 5\ \text{sec}^{-1}$,

(i) find the value of the 'amplitude' A_0 of the damped oscillatory motion at $t = 0$,

(ii) calculate the time t_1 at which the displacement first reaches its maximum X_1,

(iii) compute the maximum displacement X_1

[Use data: $\tan^{-1}(2.4) = 1.176$, $e^{-0.49} = 0.613$ and $\sin(1.176) = 0.923$]

[IIT-JAM GP : 2012]

Soln. Differential equation of one dimensional damped harmonic oscillation is described by the differential equation

$$\ddot{x} = 2\beta\dot{x} + \omega_0^2 x = 0$$

$$(D^2 + 2\beta D + \omega_0^2)x = 0$$

$$D = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega^2}}{2}$$

$$= -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$$

$$= -\beta \pm i\omega \text{ where } \omega' = \sqrt{\omega^2 - \beta^2}$$

So, $x = e^{-\beta t}[C_1 \sin \omega' t + C_2 \cos \omega' t]$

$A_0 e^{-\beta t} \sin(\omega' t + \delta)$

Velocity $V = \frac{dx}{dt}$

$$= A_0 e^{\beta t}[\omega' \cos(\omega' t + \delta) - \beta \sin(\omega' t + \delta)]$$

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At $t = 0$, $V = V_0$

$\Rightarrow A_0[\omega' \cos\delta - \beta \sin\delta] = V_0$

$A_0[\omega' \cos\delta - \beta \sin\delta] = V_0$

At $t = 0, x = 0$

$\Rightarrow A_0 \sin\delta = 0$

So, $\delta = 0$

So, $x = Ae^{-\beta t} \sin \omega' t$

$$V = \frac{dx}{dt} = A_0 e^{-\beta t}[\omega' \cos \omega' t - \beta \sin \omega' t]$$

At $t = 0$, $V = V_0$

$V_0 = A_0 \omega'$

$$\omega' = \sqrt{\omega_0^2 - \beta^2} = \sqrt{169 - 25} = 12$$

$V_0 = 0.13$

$A_0 = \frac{0.13}{12} m = 1.08\text{cm}$

At displacement maximum, $\frac{dx}{dt} = 0$

$\Rightarrow \tan \omega' t_1 = \frac{\omega'}{\beta} = \frac{12}{15} = 2.4$

$t_1 = \frac{1}{\omega}, \tan^{-1} 2.4$

$= \frac{1.176}{12} = 0.098 \text{ sec}$

Maximum displacement

$x_{max} = A_0 e^{-\beta t} \sin \omega' t$

$= 1.08\, e^{-5 \cdot 0.098} \sin 1.176$

$= 1.08\, e^{-0.49} \sin 1.176$

$= 0.611 \text{ cm}$

2.2 FORCED VIBRATION AND RESONANCE:

The natural vibration of a system dies out with time due to damping force. However, the system can be maintained in a condition of vibration by applying external periodic force. When an external periodic force acts on a vibrating system the system tends to vibrate with its own natural frequency, while the external driving force tries to impress its own frequency on the vibrating system. Initially the system vibrates with both the frequencies. However, in course of time the natural vibration dies out due to the damping forces that operate on the system and finally in the steady state the system vibrates with the frequency of the driving force. Such vibrations are called forced vibration. The driven system extracts energy from the driving system.

Resonance is a special case of forced vibration. If the frequency of the driving force agrees with the natural frequency of the system then the system oscillates with maximum amplitude or velocity. This is known as resonance or sympathetic vibration. In this case the driving force communicates maximum power to the system.

2.2.1 Mathematical analysis of forced vibration:

Suppose we consider the motion of a mechanical oscillator of mass m under the action of a restoring force proportional to displacement, a damping force proportional to its instantaneous velocity and an external periodic force represented by $F = F_0 \cos \omega t$. The equation of motion of such an oscillator is

$$m\frac{d^2x}{dt^2} = F_0 \cos \omega t - sx - R_m \frac{dx}{dt} \qquad \ldots (29)$$

where x is the displacement from equilibrium position at any instant of time t, s is the stiffness and R_m is the damping constant. The equation may be rewritten as,

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = f_0 \cos \omega t = \text{Re. } f_0 e^{j(\omega t - \phi)} \cdot e^{j\phi} \quad \ldots (30)$$

where $2b = R_m / m$, $\omega_0 = \sqrt{s/m}$, $f_0 = F_0 / m$ and 'Re' stands for the real part of the complex quantity on its right hand side. To solve the Equation (30) we note that for small damping the complementary

function (i.e., the solution with R.H.S zero) represents a damped oscillatory motion which dies out after some time. It is called transient. The particular integral represents the forced vibration. Since ultimately in the steady state the system vibrate with the frequency of the driving force

We may try a solution of the form

$$x = \text{Re}\, Ae^{j(\omega t - \phi)}$$

Here ϕ is defined as the phase by which displacement leads force.

Substituting it in equation (30), we get

$$\text{Re}\left(-\omega^2 + j\,2b\omega + \omega_0^2\right) A\, e^{j(\omega t - \phi)} = \text{Re}\, f_0 e^{j(\omega t - \phi)} \cdot e^{j\phi}$$

This is true for all 't' and hence A will be given by

$$A = \frac{f_0 \times e^{j\phi}}{\left(\omega_0^2 - \omega^2\right) + j2b\omega} \Rightarrow Ae^{-j\phi} = \frac{f_0}{\left(\omega_0^2 - \omega^2\right) + j2b\omega} \qquad \ldots (31)$$

Simplifying, we get $A = \dfrac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2}}$ and $\tan\phi = \dfrac{2b\omega}{\omega_0^2 - \omega^2}$ ... (32)

$$\Rightarrow \qquad x = \text{Re}\, Ae^{j(\omega t - \phi)} \Rightarrow x = A\cos(\omega t - \phi) \qquad \ldots (33)$$

For small damping the complementary function which is the solution of the equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0$$

is given by $x = c.e^{-bt}\cos\left(\sqrt{\omega_0^2 - b^2}\,t - \delta\right)$

where c and δ are two arbitrary constants. Hence the general solution of the Equation (30) is given by

$$x = c.e^{-bt}\cos\left(\sqrt{\omega_0^2 - b^2}\,t - \delta\right) + A\cos(\omega t - \phi) \qquad \ldots (34)$$

The first term on the right hand side of Equation (34) gives the transient and the second term represents the steady state motion.

2.2.2 Amplitude and velocity resonances:

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For certain value of ω the amplitude becomes maximum and we say there is amplitude resonance between the driver and the driven system. Now for A to be maximum, we must have

$$\frac{dA}{d\omega} = 0 \text{ and } \frac{d^2A}{d\omega^2} < 0$$

From Equation (32) it is clear that for $dA/d\omega = 0$ we must have,

$$\frac{d}{d\omega}\left\{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2\right\} = 0 \qquad \Rightarrow \qquad 2\left(\omega_0^2 - \omega^2\right)(-2\omega) + 8b^2\omega = 0$$

$$\Rightarrow \qquad \omega = \sqrt{\omega_0^2 - 2b^2} = \omega_r \,(\text{say})$$

It is easy to show that $\dfrac{d^2A}{d\omega^2} < 0$ at $\omega = \omega_r$. Thus ω_r is the frequency at which amplitude resonance occurs.

Obviously this frequency ω_r is slightly smaller than both the natural frequency ω_0 and the frequency $\sqrt{\omega_0^2 - b^2}$ of damped oscillation. The maximum displacement amplitude at resonance is obtained by putting $\omega = \sqrt{\omega_0^2 - 2b^2}$ in Equation (32).

Thus, $$A_{max} = \frac{f_0}{2b\sqrt{b^2+\omega^2}} = \frac{f_0}{2b\sqrt{\omega_0^2 - b^2}} \quad \ldots (35)$$

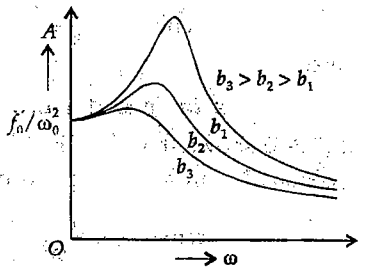


Figure 2.5: Variation of displacement amplitude with driving force frequency 'ω' for various values of b(damping factor)

Since potential energy is given by $E_p = \frac{1}{2}sx^2$, its value becomes maximum at the amplitude resonance frequency.

The instantaneous velocity of the system undergoing forced vibration may be obtained from Equation (33) as

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t - \phi) = V_0\cos(\omega t - \phi + \pi/2) \quad \ldots (36)$$

where $$V_0 = A\omega = \frac{f_0\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2}} = \frac{f_0}{\sqrt{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4b^2}} \quad \ldots (37)$$

It will be maximum for certain value of ω and we say there is velocity or energy resonance between the driver and driven systems. Now for V_0 to be maximum

$$\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 = 0 \text{ or, } \omega = \omega_0$$

Thus the velocity resonant frequency is equal to the natural frequency of the system for free oscillation. It does not depend upon the damping in the system. The maximum velocity amplitude at resonance is obtained by putting $\omega = \omega_0$ in Equation (37). Thus

$$v_{max} = \frac{f_0}{2b} \quad \ldots (38)$$

As the kinetic energy $E_K = \frac{1}{2}mv^2$, it will be maximum at velocity resonance.

Ex. 5. Consider the following forced oscillator equation:

$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + 16x = 9\cos(5t)$, where the damping coefficient β is a constant. As $\beta \to 0$, the general solution $x(t)$ is given by $x(t) = \cos(5t + \varphi) + A\cos(4t + \psi)$. The phase, φ, tends to

(a) π (b) $-\pi$ (c) 0 (d) π/2

[IIT-JAM GP : 2010]

Soln. **(a)**

As $\beta \to 0$

$(D^2 + 16)x = 9\cos 5t$

$$PI = \frac{9}{D^2 + 16}\cos 5t$$

$$= \frac{9}{-25 + 16} \cdot \cos 5t$$

$$= -\cos 5t$$

$$= \cos(5t + \pi)$$

Phase $\phi = \pi$

2.2.3 Phase relations in forced vibration:

Equation (33) shows that the displacement x lags behind the driving force, $F = F_0 \cos \omega t$, by an angle ϕ given by Equation (32) as

$$\tan\phi = \frac{2b\omega}{\omega_0^2 - \omega^2} \qquad \ldots (39)$$

Suppose ω is changed from 0 to ∞. Let us find how ϕ changes with ω:

(i) when $\omega = 0$, $\tan\phi = 0$ and hence $\phi = 0$.

(ii) when $\omega < \omega_0$, $\tan\phi$ is +ve. This indicates that $0 < \phi < \pi/2$.

(iii) when $\omega = \omega_0$, $\tan\phi = \infty$ and hence $\phi = \pi/2$.

(iv) when $\omega > \omega_0$, $\tan\phi$ is –ve, which indicates that $\pi/2 < \phi < \pi$.

(v) As $\omega \to \infty$, $\tan\phi \to -0$ which indicates that $\phi \to \pi$.

From equation (39), $\sec^2\phi \dfrac{d\phi}{d\omega} = \dfrac{2b}{\omega_0^2 - \omega^2} - \dfrac{2b\omega}{\left(\omega_0^2 - \omega^2\right)^2}(-2\omega)$

$$\Rightarrow \qquad \left(1 + \tan^2\phi\right)\frac{d\phi}{d\omega} = \frac{2b\left(\omega_0^2 + \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2}$$

$$\Rightarrow \qquad \frac{d\phi}{d\omega} = \frac{1}{1 + \dfrac{4b^2\omega^2}{\left(\omega_0^2 - \omega^2\right)^2}} \cdot \frac{2b\left(\omega_0^2 + \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2} = \frac{2b\left(\omega_0^2 + \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + 2b^2\omega^2}$$

At velocity resonance $\omega = \omega_0$ and then $\dfrac{d\phi}{d\omega} = \dfrac{2}{b}$ ... (40)

So smaller is the damping factor b greater is the rate of change of phase angle near resonance.

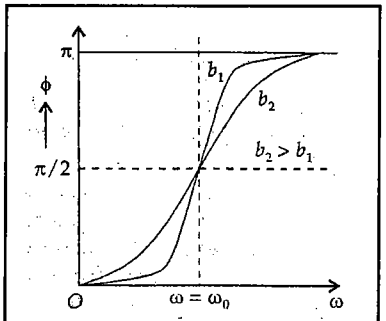


***Figure* 2.6 (a):** Phase relationship between displacement and driving force.

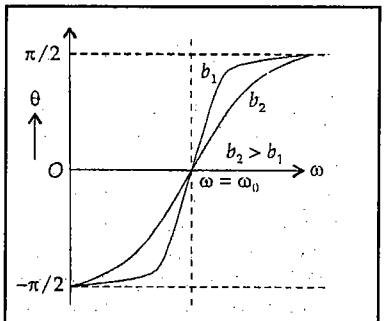


***Figure* 2.6 (b):** Phase-relationship between and driving force ($\theta = \phi - \pi/2$)

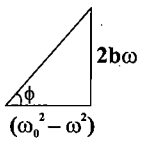
2.2.4 Energy in forced vibration:

The instantaneous rate of work done by the driving force i.e., the instantaneous power supplied to the oscillator by the driving force is $P = F_0 \cos\omega t . \frac{dx}{dt} = -F_0 A\omega \cos\omega t . \sin(\omega t - \phi)$

$$= -\frac{F_0 A\omega}{2}\left[\sin(2\omega t - \phi) - \sin\phi\right] \qquad \ldots (41)$$

The average power over a complete period is

$$< P > = \frac{1}{T}\int_0^T P.dt = \frac{F_0 A\omega}{2}.\sin\phi \left[\because \int_0^T \sin(2\omega t - \phi)dt = 0\right]$$



$$= \frac{F_0 A\omega}{2}.\frac{2b\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2}} = mb\omega^2 A^2 \qquad \ldots (42)$$

Now the rate of work done against damping force is

$$\frac{dW}{dt} = R_m \frac{dx}{dt}.\frac{dx}{dt} = R_m \omega^2 A^2 \sin^2(\omega t - \phi) = \frac{R_m \omega^2 A^2}{2}\left[1 - \cos 2(\omega t - \phi)\right]$$

Averaging over a complete period we get

$$\left\langle \frac{dW}{dt} \right\rangle = \frac{1}{2} R_m \omega^2 A^2 = mb\omega^2 A^2 \qquad \ldots (43)$$

Thus, we prove that in the steady state the average power supplied to the oscillator by the driving force is just equal to that being dissipated by the damping forces.

2.2.5 Sharpness of resonance and quality factor:

The time average power, from Equation (43), is

$$\langle P \rangle = mb\omega^2 A^2 = \frac{mb\omega^2 f_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2}$$

$$= \frac{mbf_0^2}{\left(\frac{\omega_0^2}{\omega} - \omega\right) + 4b^2} \qquad \ldots (44)$$

$< P >$ is maximum when $\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 = 0$ or, $\omega = \omega_0$.

Putting $\omega = \omega_0$ in Equation (44) we get the maximum average power at resonance as

$$< P >_{max} = \frac{mbf_0^2}{4b^2} \qquad \ldots (45)$$

Therefore, $$< P > = < P >_{max} \frac{4b^2\omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4b^2\omega^2} \qquad \ldots (46)$$

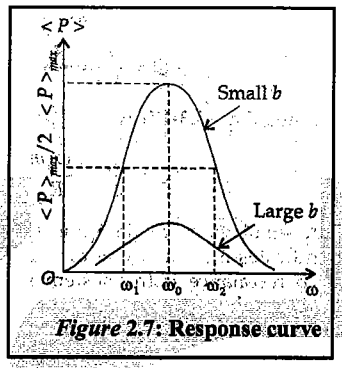


Figure 2.7: **Response curve**

At half power points, $< P > = \frac{< P >_{max}}{2}$

$$\Rightarrow \frac{mb\omega^2 f_0^2}{(\omega_0^2 - \omega^2)^2 + 4b^2\omega^2} = \frac{1}{2}\frac{mbf_0^2}{4b^2}$$

$$\Rightarrow (\omega_0^2 - \omega^2)^2 + 4b^2\omega^2 = 8\omega^2 b^2$$

$$\Rightarrow \omega_0^2 - \omega^2 = \pm 2b\omega$$

$$\Rightarrow \omega^2 - \omega_0^2 \pm 2b\omega$$

$$\Rightarrow \omega = \mp b \pm \sqrt{b^2 + \omega_0^2} \qquad \ldots (47)$$

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These are the two quadratic equation $\omega^2 - \omega_0^2 + 2b\omega = 0$ and $\omega^2 + \omega_0^2 + 2b\omega = 0$

Since negative frequency are not allowed, we take the position roots only

$$\omega_1 = -b + \sqrt{b^2 + \omega_0^2} \text{ and } \omega_2 = b + \sqrt{b^2 + \omega_0^2} \qquad \ldots (48)$$

Thus bandwidth, $\Delta\omega = \omega_2 - \omega_1 = 2b$

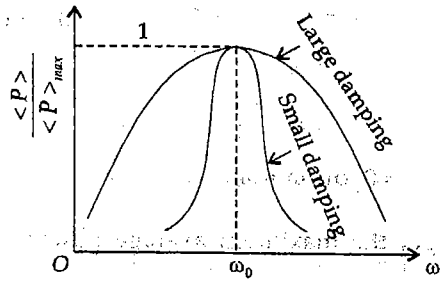

***Figure* 2.8: Dependence of sharpness of resonance on damping**

Quantitatively the sharpness of resonance is measured in terms of quality factor Q which is defined as

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\text{Resonant frequency}}{\text{Bandwidth}} \Rightarrow Q = \frac{\omega_0}{2b} = \frac{\omega_0 m}{R_m} \qquad \ldots (49)$$

A small value of damping constant R_m makes Q large. Again a large Q indicates a small value of bandwidth i.e., a sharp resonance. In case of sharp resonance if the system is driven by a force consisting of a large number of harmonic components then it will give significant response only to the components having frequencies close to the resonant frequency. The system thus possesses selective properties. This is known as selectivity.

SOLVED EXAMPLES

1. A mass of 0.01 kg is acted upon by a restoring force $0.01\,N.m^{-1}$ and a resisting force of 2×10^{-3} $N.s.n^{-1}$. Find out whether the motion is oscillatory or non-oscillatory. Also, find the value of resistive force for the motion to be critically damped.

Soln. Here $m = 0.01$ kg, $s = 0.01\,N.m^{-1}$ and $R_m = 2 \times 10^{-3}\,N.s.m^{-1}$

So, $b = \frac{R_m}{2m} = \frac{2\times10^{-3}}{2\times0.01} = 0.1\,N.s.m^{-1}.kg^{-1}$ $\omega_0 = \sqrt{\frac{s}{m}} = \sqrt{\frac{0.01}{0.01}} = 1$

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As $b < \omega_0$, $\sqrt{b^2-\omega^2} < 1$ or $i\sqrt{\omega^2-b^2}$ the motion is oscillatory.

For the motion to be critically damped. $b = \omega_0$

$$\Rightarrow \quad \frac{R_m}{2m} = \sqrt{s/m}$$

$$\Rightarrow \quad R_m = 2\sqrt{sm} = 2\sqrt{0.01\times0.01} = 0.02\,N.s.m^{-1}$$

2. A mass m = 0.2 kg is suspended by a light spring of stiffness constant $20\,N.m^{-1}$. The energy associated with the vertical oscillation of the system is observed to decay to 1/e of initial value in 50 s. Assuming the damping force $(R_m v)$ proportional to the velocity (v), find the damping force constant R_m. Also, find the Q-value of the oscillator and the percentage change in frequency due to damping.

Soln. The energy decays according to the law, $E = E_0 e^{-2bt}$

At $t = 50$ sec, $E = \frac{E_0}{e} = E_0 e^{-2bt} \Rightarrow e = e^{2bt}$

$\Rightarrow 1 = 2bt \Rightarrow b = \frac{1}{2t}$

Thus, $R_m = 2bm = \frac{m}{t} = \frac{1}{50} \times 0.2 = 4 \times 10^{-3}\ N.s.m^{-1}$

The undamped natural frequency

$$\omega_0 = \sqrt{\frac{s}{m}} = \sqrt{\frac{20}{0.2}} = 10\ rad s^{-1}$$

$$\Rightarrow \quad Q = \frac{\omega}{2b} = \frac{1}{2b}\sqrt{\omega_0^2 - b^2} = 50\sqrt{10^2 - \frac{1}{10^4}} = 500$$

Now, $\omega = \sqrt{\omega_0^2 - b^2} = \omega_0\left(1 - \frac{b^2}{\omega_0^2}\right)^{1/2} = \omega_0\left(1 - \frac{b^2}{2\omega_0^2}\right) = \omega_0\left(1 - \frac{1}{8Q^2}\right)\left[\because Q = \frac{\omega_0}{2b}\right]$

$$\Rightarrow \quad \frac{\omega_0 - \omega}{\omega_0} = \frac{1}{8Q^2}$$

Therefore, the percentage change in frequency due to damping is

$$\frac{\omega_0 - \omega}{\omega_0} \times 100 = \frac{100}{8Q^2} = \frac{100}{8 \times (500)^2} = 5 \times 10^{-4}\%$$

3. A body of mass $m = 10$ kg is subjected to a restoring force of stiffness constant $s = 10\ N.m^{-1}$. The system is subjected to a damping force proportional to its velocity with the constant of proportionality as $R_m = 10\ N.s.m^{-1}$. The mass is given an initial impulse of $0.68\ kg.m.s^{-1}$ when at rest. Find the subsequent displacement of the mass as a function of time.

Soln. The equation of motion of the mass is $m\ddot{x} = -R_m\dot{x} - sx$

and its solution is of the form

$$x = e^{-bt}(A\cos\omega t + B\sin\omega t)$$

where $b = \frac{R_m}{2m}$, $\omega = \sqrt{\omega_0^2 - s/m}$; A and B are constants to be determined from given initial conditions.

At $t = 0, x = 0$. This indicates that $A = 0$.

$$\Rightarrow \quad x = Be^{-bt}\sin\omega t$$

Now the impulse is equal to change of momentum. Therefore

$$m\dot{x} - 0 = 0.68$$

Therefore, Initial velocity $\dot{x}_0 = \frac{0.68}{10} = 0.068\ ms^{-1}$

$$b = \frac{R_m}{2m} = \frac{10}{2 \times 10} = 0.5 s^{-1}, \omega_0$$

$$= \sqrt{s/m} = 1\ rad.s^{-1}, \omega = \sqrt{\omega_0^2 - b^2} = 0.866$$

$$\dot{x} = -bBe^{-bt}\sin\omega t + \omega Be^{-bt}\cos\omega t$$

At $t = 0, 0.068 = \omega B$.

$$\Rightarrow \qquad B = \frac{0.068}{0.866} = 0.078\,\text{m}$$

Thus $x = 0.078e^{-0.5t}\sin 0.866t$

This equation describes the subsequent motion.

4. In a damped harmonic oscillator the amplitude of vibration decreases from10 cm to 1 cm after 100 complete cycles of oscillation. The time period of oscillation is 2.3 secs. Estimate the first undamped amplitude.

Soln. Log decrement $\lambda = \frac{1}{n}\log_e \frac{A_1'}{A_{n+1}'} = \frac{1}{2\times 100}\log_e \frac{10}{1} = 0.0115$, where n is the number of the half cycles.

Therefore, Undamped amplitude $A = A_1(1 + \lambda/2) = 10\left(1 + \frac{0.0115}{2}\right) = 10.06\,\text{cm}$

5. Show that in case of a forced vibration under a simple harmonic force the half-power frequencies correspond to the phase angle range given by $\tan\phi = \pm 1$ where ϕ is the phase difference between the displacement and the driving force.

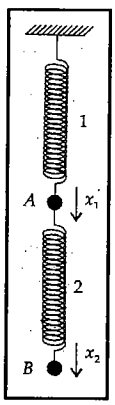
Soln. Average power $< P > = \dfrac{mbf_0^2}{\left(\frac{\omega_0^2}{\omega} - \omega\right) + 4b^2}$

At half power points, $\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 = 4b^2$ or, $\omega_0^2 - \omega^2 = \pm 2b\omega$

Therefore the phase angle ϕ is given by $\tan\phi = \dfrac{2b\omega}{\omega_0^2 - \omega^2} = \pm\dfrac{(\omega_0^2 - \omega^2)}{\omega_0^2 - \omega^2} = \pm 1$.

6. Two equal masses m are attached with two identical massless springs of stiffness a as indicated in Fig. below. Find the normal frequencies and the ratios of amplitudes of the normal modes.

Soln. Let x_1 and x_2 be the instantaneous displacements of the masses from their equilibrium positions. If $x_2 > x_1$ then the spring 2 is extended and the tension $s(x_2 - x_1)$ in the spring will act in favour of acceleration $\ddot{x}_1$ but against the acceleration $\ddot{x}_2$. Thus the equations of motion are



$$m\ddot{x}_1 = -sx_1 + s(x_2 - x_1)$$

$$m\ddot{x}_2 = -s(x_2 - x_1)$$

Putting $s/m = \omega_0^2$ and rearranging we get

$$\ddot{x}_1 + 2\omega_0^2 x_1 - \omega_0^2 x_2 = 0 \qquad \ldots. (1)$$

and $$\ddot{x}_2 + \omega_0^2 x_2 - \omega_0^2 x_1 = 0 \qquad \ldots. (2)$$

In a normal mode both the masses oscillate with the same frequency. To find the normal frequencies we let

$$x_1 = A_1 e^{j\omega t} \text{ and } x_2 = A_2 e^{j\omega t}$$

Substituting these solution in Equation (1) and (2) we get

$$(2\omega_0^2 - \omega^2)A_1 \omega_0^2 A_2 = 0 \qquad \ldots. (3)$$

$$-\omega_0^2 A_1 + \left(\omega_0^2 - \omega^2\right) A_2 = 0 \qquad \ldots. (4)$$

For non-trivial solution, $\begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0 \quad \Rightarrow \quad \omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$

Therefore the normal frequencies are $\omega_1 = \left(\frac{3-\sqrt{5}}{2}\right)^{1/2} \omega_0$ and $\omega_2 = \left(\frac{3+\sqrt{5}}{2}\right)^{1/2} \omega_0$

Putting $\omega = \omega_1$ in Equation (3) and (4) we get $\frac{A_1}{A_2} = -\frac{\sqrt{5}-1}{2}$

It is the ratio of amplitudes for the normal mode of frequency $\omega = \omega_1$.

Putting $\omega = \omega_2$ in Equations (3) and (4) we get $\frac{A_1}{A_2} = -\frac{\sqrt{5}+1}{2}$

It is the ratio of amplitudes for the normal mode of frequency $\omega = \omega_2$.

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Chapter 3

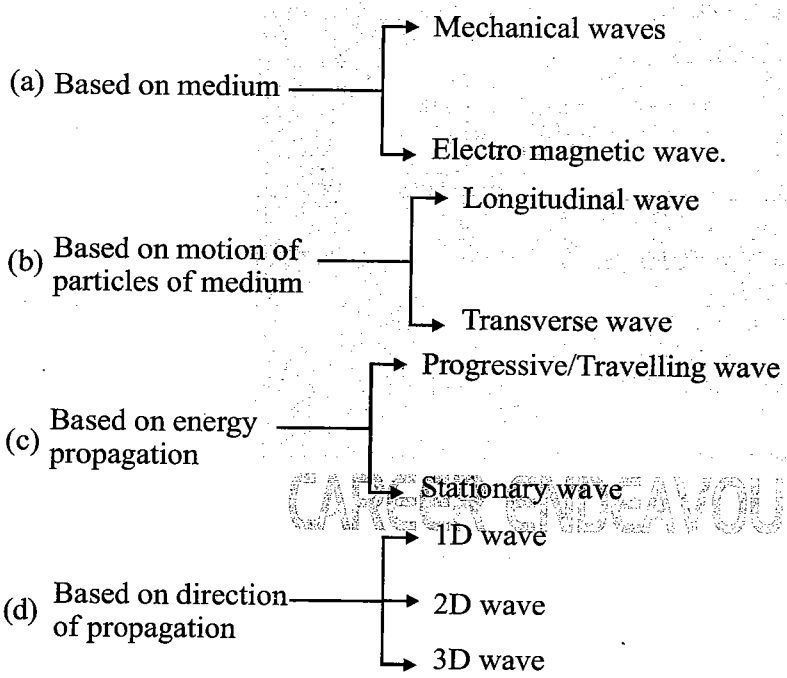
Acoustic Wave Motion

3.1 WAVES:

A wave is a disturbance which propagate in space, transport energy from one place to another without transport of matter.

Example: Ripples on a pond, sound we hear etc.

3.1.1 Classification of wave:



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3.1.2 Wave Equation:

Let us consider the equation representing a plane progressive harmonic wave

$$\psi(x,t) = a\sin(\omega t - kx) \quad \ldots (1)$$

$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2 a\sin(\omega t - kx) \quad \ldots (2)$$

and,

$$\frac{\partial^2\psi}{\partial x^2} = -k^2 a\sin(\omega t - kx) \quad \ldots (3)$$

$$\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2} \quad \ldots (4)$$

where $c = \omega/k$ is the velocity of the wave. Equation (4) is the well-known one dimensional differential wave equation.

Ex. 1. Which of the following CANNOT be a solution of the one dimensional wave equation:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

(a) $A\cos k(x+ct)$ (b) $A\exp[ik(x-ct)]$

(c) $A\exp\left[-\alpha(x-ct)^2\right]$ (d) $\frac{A}{(x^2-c^2t^2)}$

[IIT-JAM GP : 2010]

Soln. **(d)** The solution must satisfy the given partial differential equation and only (d) does not satisfy. Also, only those expression which contain (ax ± bt) term only satisfy wave equation.

Ex. 2. Two waves described by $y_1 = A\sin(\omega t + kx)$ and $y_2 = A\sin(\omega t - kx)$ are travelling along a string. Let $A = 0.001m$, $k = 3.142\ m^{-1}$ and $\omega = 157.1\ s^{-1}$ (Take $\pi = 3.142$)
(a) Find the magnitude and direction of velocity of these waves.
(b) What shall be the amplitude of resultant wave on the string at $x = 0.5m$.

[IIT-JAM GP : 2006]

Soln. Waves is described by following equations

(a) $y_1 = A\sin(\omega t + kx)$ and $y_2 = A\sin(\omega t - kx)$

Velocity of each waves are $v = \frac{\omega}{k} = \frac{157.1}{3.142} = 50$ m/s

First wave is going towards negative axis and second towards positive x-axis.

Thus, $\vec{v}_1 = -50\hat{i}$ m/s, and $\vec{v}_2 = \frac{\omega}{k} = 50\hat{i}$ m/s.

(b) The resultant waves is described by

$$\begin{aligned} y &= y_1 + y_2 \\ &= A\sin(\omega t + kx) + A\sin(\omega t - kx) \\ &= 2A\sin\omega t\cos kx \\ &= 2(A\cos kx)\sin\omega t \end{aligned}$$

The amplitude of resultant wave $= 2A\cos kx$

At $x = 0.5\ m$

Amplitude $= 2\times 0.001\cos(3.142\times 0.5) = 0.002\cos\frac{\pi}{2} = 0.$

Ex. 3. A plane wave of frequency ω propagates so that planes of constant phase move with speeds v_1, v_2 and v_3 along the x, y and z axes, respectively. The wave vector, K of the plane wave (in terms of the Cartesian unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$) is:

(a) $K = \omega\left(\frac{1}{v_1}\hat{i} + \frac{1}{v_2}\hat{j} + \frac{1}{v_3}\hat{k}\right)$ (b) $K = \omega\left(\hat{i}+\hat{j}+\hat{k}\right)/\sqrt{v_1^2+v_2^2+v_3^2}$

(c) $K = \omega\left(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}\right)/\left(v_1^2+v_2^2+v_3^2\right)$ (d) $K = \left(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}\right)/\omega$

[JNU : 2010]

Soln. **(c)** $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

Wave vector has magnitude equal to ω/v and direction is along the direction of the propagation.

$$\vec{k} = \frac{\omega}{|\vec{v}|}(\hat{v}) = \frac{\omega}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \frac{\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)}{\sqrt{v_1^2 + v_2^2 + v_3^2}}$$

$$\boxed{\vec{k} = \frac{\omega\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)}{v_1^2 + v_2^2 + v_3^2}}$$

3.2 TRAVELLING WAVES:

It progresses with fixed velocity in a medium.
Example: Waves on liquid surface.

3.2.1 Equation of travelling waves:

(i) In + X direction: $y = a\sin\omega\left(t - \frac{x}{v}\right)$ ... (5)

(ii) In –X direction: $y = a\sin\omega\left(t + \frac{x}{v}\right)$ ... (6)

Ex. 4. A plane harmonic light wave travelling through a medium is represented (in S1 system) by

$$E_x(y,t) = E_0 \sin 2\pi\left(\frac{y}{3\times10^{-7}} - 5\times10^{14}t\right)$$

The refractive index of the medium at this frequency will be
(a) 1.3 (b) 1.5 (c) 2.0 (d) 2.5

[IIT-JAM GP : 2009]

Soln. (c)
A plane wave travelling along y-axis is given by
$E_x(y,t) = E_0 \sin(ky - \omega t)$
Comparing with given equation in problem, we get

Wavevector $k = \frac{2\pi}{3\times10^{-7}}$, and

angular frequency $\omega = 2\pi \times 5\times10^{14}$

Velocity $v = \frac{\omega}{k} = 5\times10^{14}\times3\times10^{-7}$

$= 1.5\times10^{8}$

Refractive index

$$\mu = \frac{c}{v} = 2$$

Ex. 5. If a function of distance x and time t is given by $f(x - vt)$, then it represents

(a) a wave travelling along $+x$-direction and satisfies the wave equation $\frac{\partial f}{\partial x} + \frac{1}{v}\frac{\partial f}{\partial t} = 0$

(b) a wave travelling along $+x$-direction and satisfies the wave equation $\frac{\partial f}{\partial x} - \frac{1}{v}\frac{\partial f}{\partial t} = 0$

(c) a wave travelling along $-x$-direction and satisfies the wave equation $\frac{\partial f}{\partial x} + \frac{1}{v}\frac{\partial f}{\partial t} = 0$

(d) a wave travelling along $-x$-direction and satisfies the wave equation $\frac{\partial f}{\partial x}-\frac{1}{v}\frac{\partial f}{\partial t}=0$

[IIT-JAM GP : 2008]

Ans. **(a)**

Given wave function $f(x-vt)$

Now, $\frac{\partial f}{\partial x}(x-vt)=\dot{f}(x-vt)$ and $\frac{\partial f}{\partial t}(x-vt)=-v\dot{f}(x-vt)$

So, $\frac{\partial f}{\partial x}(x-vt)+\frac{1}{v}\frac{\partial f}{\partial t}(x-vt)=\dot{f}(x-vt)-\frac{v}{v}\dot{f}(x-vt)=0$

Also, for travelling wave, phase $= x-vt =$ constant

$\Rightarrow \frac{dx}{dt}-v=0 \Rightarrow \frac{dx}{dt}=+v$

i.e., wave travel along positive x-axis.

So, option (a) is correct.

Ex. 6. A traveling pulse is given by $f(x,t)=A\exp\left(\frac{2abxt-a^2x^2-b^2t^2}{c^2}\right)$, where A, a, b and c are positive constants of appropriate dimensions. The speed of the pulse is

(a) $\frac{b}{a}$ (b) $\frac{2b}{a}$ (c) $\frac{cb}{a}$ (d) $\frac{b}{2a}$

[IIT-JAM : 2013]

Soln. **(a)** A travelling pulse is given by

$$f(x)=A\exp\left(\frac{2abxt-a^2x^2-b^2t^2}{c^2}\right)=A\exp\left[-\frac{(a^2x^2+b^2t^2-2abxt)}{c^2}\right]=A\exp\left[-\frac{(ax-bt)^2}{c^2}\right]$$

$$=A\exp\left[-\left(\frac{a}{c}x-\frac{b}{c}t\right)^2\right]=Ae^{-\left(\frac{a}{c}x-\frac{b}{c}t\right)^2}$$

$$\boxed{v=\frac{\omega}{k}=\frac{\text{coeff of } t}{\text{coeff of } x}}=\frac{b/c}{a/c}=\frac{b}{a}$$

Ex. 7. Consider the solution to the one-dimensional wave equation: $\Psi(x,t)=\Psi_0\exp i(3x+147t)$, where x is in meters and t in seconds. The velocity of the wave is

(a) 7 m/s (b) –1/7 m/s (c) –7 m/s (d) 1/7 m/s

[IISc : 2009]

Soln. None

$\psi(x,t)=\psi_0\exp[i(3x+147t)]$

Given wave travel in negative x-axis became coefficient of x and t, both are positive.

Velocity of wave $v=\frac{\omega}{k}=-\frac{147}{3}=-49$ m/s

Ex. 8. Consider the following four functions each representing a plane wave :

(I) $f_1(x,y,z,t)=\sin\left(\sqrt{222}t-2x-7y+13z\right)$

(II) $f_2(x,y,z,t)=\sin\left(\sqrt{14}t-2x-3y-z\right)$

(III) $f_3(x,y,z,t)=\sin\left(\sqrt{89}t-4x-3y+8z\right)$

(IV) $f_4(x,y,z,t)=\sin\left(\sqrt{125}t+5x-6y+8z\right)$

Which of the following pairs represents waves moving in mutually perpendicular directions ?

(a) I and III (b) 1 and IV (c) II and III (d) II and IV

[JNU : 2011]

Sln. **(d)** Comparing with standard equation

$f=A\sin\left(\omega t-\vec{k}.\vec{r}\right)$, we get

$\vec{k}_1=2\hat{i}+7\hat{j}-13\hat{k}\;;\;\vec{k}_2=2\hat{i}+3\hat{j}+\hat{k}\;;\;\vec{k}_3=4\hat{i}+3\hat{j}-8\hat{k}\;;\;\vec{k}_4=-5\hat{i}+6\hat{j}-8\hat{k}$

Clearly, $\vec{k}_2.\vec{k}_4=0$

Therefore, wave II and IV are perpendicular to each other.

Ex. 9. Two vibrating strings of lengths L and 3L and radii 3R and R, respectively, are stretched under the same tension. Both the strings vibrate in their fundamental modes, the one of length L with frequency ν_1 and the other with frequency ν_2. If the two strings are made of same material then the ratio ν_1/ν_2 is given by

(a) 1 (b) $3^{1/2}$ (c) 3 (d) 9 **[DU 2015]**

Sln. (a) Since both the strings are made of same material, the density of the material is same for both.

Let ρ be the density.

Now, μ_1 = mass unit length of string

$$=\frac{\rho\pi(3R)^2L}{L}=9\pi\rho R^2$$

(Since, length$_1$ = L and radius$_1$ = 3R)

Similarly, $\mu_2=\dfrac{\rho\pi R^2(3L)}{3L}=\pi\rho R^2$

(Since, length$_2$ = 3L and radius$_2$ = R)

Now, frequency $\nu=\dfrac{1}{2L}\sqrt{\dfrac{T}{\mu}}$

Where, ν = frequency of wave

L = length of string

T = Tension in string

μ = mass per unit length of string)

$$\therefore\quad \nu_1=\frac{1}{2L}\sqrt{\frac{T}{9\pi\rho R^2}}\quad\&\quad \nu_2=\frac{1}{6L}\sqrt{\frac{T}{\pi\rho R^2}}$$

$$\Rightarrow\quad \frac{\nu_1}{\nu_2}=\frac{1}{2L}\frac{1}{3}\sqrt{\frac{T}{\pi\rho R^2}}\times 6L\sqrt{\frac{\pi\rho R^2}{T}}=1$$

Ex. 10. A transverse wave is described by the equation $y = y_0 \sin 2\pi\left(ft - \frac{x}{\lambda}\right)$. The maximum particle velocity is equal to four times the wave velocity if

(a) $\lambda = \pi$ (b) $\lambda = \pi y_0/2$ (c) $\lambda = \pi$ (d) $\lambda = 2\pi y_0$

[IIT-JEE : 1984]

Soln. **(b)**

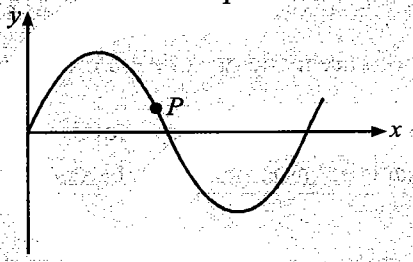
Wave velocity $v = \frac{\text{coefficient of } t}{\text{coefficient of } x} = \frac{2\pi f}{2\pi/\lambda} = \lambda f$

Maximum particle velocity $v_{pm} = \omega A = 2\pi f y_0$, $[\omega = 2\pi f]$

Given, $v_{pm} = 4v$ or $2\pi f y_0 = 4\lambda f$

$\therefore \lambda = \frac{\pi y_0}{2}$

Ex. 11. A transverse sinusoidal wave moves along a string in the positive x-direction at a speed of 10 cm/s. The wavelength of the wave is 0.5 m and its amplitude is 10 cm. At a particular time t, the snap-shot of the wave is shown in figure. The velocity of point P when its displacement is 5 cm is



(a) $\frac{\sqrt{3}\pi}{50}\hat{\mathbf{j}}$ m/s (b) $-\frac{\sqrt{3}\pi}{50}\hat{\mathbf{j}}$ m/s (c) $\frac{\sqrt{3}\pi}{50}\hat{\mathbf{i}}$ m/s (d) $-\frac{\sqrt{3}\pi}{50}\hat{\mathbf{i}}$ m/s

[IIT-JEE : 2008]

Soln. **(a)**

Consider a wave $y(x, t) = A\sin(kx - \omega t)$

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Slope $\frac{dx(x,t)}{dx,t} = Ak\cos(kx - \omega t)$

$= Ak\left[1 - \sin^2(kx - \omega t)\right]^{1/2}$

$= Ak\left[1 - \left(\frac{x(x,t)}{A}\right)^2\right]^{1/2}$

At $y = 5\,\text{cm}$

$\text{Slope} = 10\,\text{cm}\left(\frac{2\pi}{0.5\times10^2\,\text{cm}}\right)\left[1 - \left(\frac{5\,\text{cm}}{10\,\text{cm}}\right)^2\right]^{1/2} \quad \left[\because k = \frac{2\pi}{\lambda}\right]$

$= 0.2 \cdot 2\pi \cdot \frac{\sqrt{3}}{2} = \frac{\pi\sqrt{3}}{5}.$

Velocity of the particle at any position is given by

$$|v_p| = |v \cdot \text{slope}| = 10\,\text{cm/s} \times \frac{\pi\sqrt{3}}{5}$$

$$= \frac{\pi\sqrt{3}}{5} \times 10 \times 10^{-2}\ \text{m/s}$$

Particle velocity $v_p = -v$ (slope of y-x graph)

Here, v = positive, as the wave is travelling in positive x-direction.

Slope at P is negative. Therefore, velocity of particle is in positive y(or $\hat{\mathbf{j}}$) direction.

$$= \frac{\pi\sqrt{3}}{50}\ \text{m/s along } \hat{j}.$$

Ex. 12. The speed of sound propagation in air as a function of temperature T is given by $v = \alpha T$, where α is a constant of appropriate dimensions. Calculate the time taken for a sound wave to travel a distance L between two points A and B, if the air temperature between the points varies linearly from T_1 to T_2.

Soln. It is given that air temperature varies linearly. Therefore, we can assume that at a distance 'x' from A temperature is given as

$$T = a + \beta x$$

At A, $x = 0$, $T = T_1$

$$\therefore \quad T_1 = a + \beta \times 0 \Rightarrow a = T_1$$

At B, $x = L$, $T = T_2$

$$\therefore \quad T_2 = T_1 + \beta L \Rightarrow \beta = \frac{T_2 - T_1}{L}$$

$$\therefore \quad T = T_1 + \frac{T_2 - T_1}{L} x \qquad \ldots (i)$$

Given, $v = \alpha T$

$$\therefore \quad \frac{dx}{dt} = \alpha\left[T_1 + \frac{T_2 - T_1}{L} x\right]$$

$$\therefore \quad \int_0^L \frac{dx}{T_1 + \frac{T_2 - T_1}{L} x} = \int_0^t \alpha\, dt \Rightarrow \frac{L}{T_2 - T_1}\left[\log\left(T_1 + \frac{T_2 - T_1}{L} x\right)\right]_0^L = \alpha t$$

$$\frac{L}{T_2 - T_1} \cdot \log\frac{T_2}{T_1} = \alpha t \quad \therefore \quad t = \frac{L}{\alpha(T_2 - T_1)} \log\frac{T_2}{T_1}$$

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3.2.2 Energy Distribution in Plane Progressive (travelling) Sound Waves:

Suppose a plane, harmonic longitudinal wave is moving through a medium along +ve x-direction. The displacement of a particle at the position x at any time t may be represented by

$$\xi(x,t) = a\cos(\omega t - kx) \qquad \ldots (7)$$

where 'ω' is the angular frequency, k is the propagating constant and a is the amplitude of vibration. Let us now consider an infinitesimally thickness δx and of unit cross-section. The layer is considered so thin that all the particles within the layer may be assumed to possess the same velocity $\partial\xi / \partial t$. The mass of this layer is $\rho_0 \delta x$ where ρ_0 is the equilibrium density.

Therefore, kinetic energy of the layer is

$$\Delta E_k = \frac{1}{2}(\rho_0 \delta x)\left(\frac{\partial \xi}{\partial t}\right)^2 \quad \ldots (8)$$

Using equation(7) we get the kinetic energy per unit volume as

$$E_k = \frac{1}{2}\rho_0 a^2 \omega^2 \sin^2(\omega t - kx) \quad \ldots (9)$$

Now the time average value of $\sin^2(\omega t - kx)$ over a complete time period (or its space average value over a complete wavelength λ) is 1/2. Thus average kinetic energy per unit volume becomes

$$\bar{E}_k = \frac{1}{4}\rho_0 a^2 \omega^2 \quad \ldots (10)$$

During the transmission of the acoustic wave the volume element is periodically compressed and rarefied. So there is change in potential energy associated with this volume change. Suppose an increase of pressure from P_0 to $P_0 + p$ compresses unit volume of the medium to a volume (1-s) where $s = \partial\xi / \partial x$ is the condensation. The potential energy associated with this volume is equal to the work done in the process of compression. Thus potential energy per unit volume is given by

$$E_p = \int_0^s p\, ds \quad \ldots (11)$$

Now by definition bulk modulus

$$B = \frac{p}{s} \text{ or, } p = Bs \quad \ldots (12)$$

Therefore, $E_p = B\int_0^s s\, ds = \frac{1}{2}Bs^2 = \frac{1}{2}B\left(-\frac{\partial \xi}{\partial x}\right)^2 \quad \ldots (13)$

Using equation (7) we get

$$E_p = \frac{1}{2}Ba^2k^2 \sin^2(\omega t - kx) \quad \ldots (14)$$

Now the time or space average value of $\sin^2(\omega t - kx)$ is 1 / 2. Therefore the average value of potential energy per unit volume becomes

$$\bar{E}_p = \frac{1}{4}Ba^2k^2 = \frac{1}{4}\rho_0 a^2 \omega^2 \quad \left[\because c = \sqrt{B/\rho_0} = \omega/k\right] \quad \ldots (15)$$

Therefore, energy density is $\bar{E} = \bar{E}_k + \bar{E}_p = \frac{1}{2}\rho_0 a^2 \omega^2 \quad \ldots (16)$

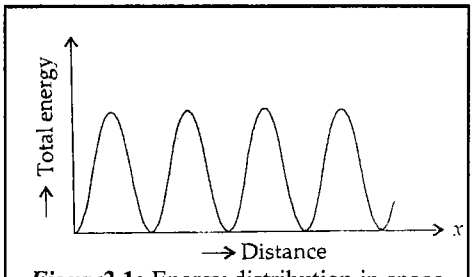


***Figure*3.1:** Energy distribution in space for a sound wave in a fluid

Ex. 13. The ends of a stretched wire of length L are fixed at $x = 0$ and $x = L$. In one experiment the displacement of the wire is $y_1 = A\sin\left(\frac{\pi x}{L}\right)\sin \omega t$ and energy is E_1 and in other experiment its displacement is $y_2 = A\sin\left(\frac{2\pi x}{L}\right)\sin 2\omega t$ and energy is E_2. Then **[IIT-JEE : 2001]**

(a) $E_2 = E_1$ (b) $E_2 = 2E_1$ (c) $E_2 = 4E_1$ (d) $E_2 = 16E_1$

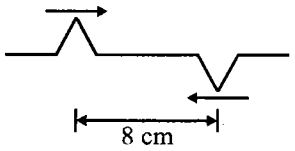
Soln. (c)

Energy $E \propto$ (amplitude)^2 (frequency)^2

Amplitude (A) is same in both the cases, but frequency 2ω in the second case is two times the frequency (ω) in the first case.

Therefore, $E_2 = 4E_1$

Ex. 14. Two pulses in a stretched string, whose centres are initially 8 cm apart, are moving towards each other as shown in the figure. The speed of each pulse is 2 cm/s. After 2 s the total energy of the pulses will be



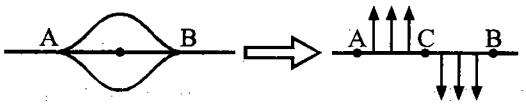
(a) zero (b) purely kinetic

(c) purely potential (d) partly kinetic and partly potential

[IIT-JEE : 2001]

Soln. (b)

After two seconds both the pulses will move 4 cm towards each other. So, by their superposition, the resultant displacement at every point will be zero. Therefore, total energy will be purely in the form of kinetic. Half of the particles will be moving upwards and half downwards.



3.3 STATIONARY WAVES OR STANDING WAVES:

3.3.1 Introduction:

When two progressive waves of the same frequency travelling in opposite directions superpose on each other, the resulting wave pattern does not propagate in either direction but remains confined within the region of overlap. Such waves are called stationary waves or standing waves. Suppose we consider two progressive waves of the same amplitude and frequency but propagating along the x-axis in opposite directions. Let these waves be represented by

$$\xi_1 = a\cos(\omega t - kx) \text{ and } \xi_2 = a\cos(\omega t + kx) \quad \ldots (17)$$

where ξ_1 and ξ_2 are the particle displacements associated with the two waves. The resultant displacement is given by the principle of superposition as

$$\xi = \xi_1 + \xi_2 = 2a\cos\omega t.\cos kx = A\cos\omega t \quad \ldots (18)$$

where $A = 2a\cos kx$.

3.3.2 Nodes and Antinodes:

Equation (18) represents a simple harmonic motion of the same frequency as the component waves but of

amplitude A which is a periodic function of the position x of the particle. Equation (18) does not represent a progressive wave because the phase angle does not contain both x and t. At places where x is such that $\cos kx = \pm 1$ the amplitude becomes maximum and has a value 2a. These points of maximum amplitude are called displacement antinodes. These positions are given by

$$kx = n\pi \qquad \text{where } n = 0, 1, 2, 3, \text{ etc.}$$

$$\Rightarrow \quad \frac{2\pi}{\lambda}x = n\pi \quad \Rightarrow \quad x = n\frac{\lambda}{2} \qquad \ldots (19)$$

The distance between two successive antinodes is $(n+1)\frac{\lambda}{2} - n\frac{\lambda}{2} = \frac{\lambda}{2}$. The amplitude A is zero at points where $\cos kx = 0$. Such points are given by

$$kx = (2n+1)\frac{\pi}{2} \qquad \text{where } n = 0, 1, 2, 3, \text{ etc.}$$

$$\Rightarrow \quad x = (2n+1)\frac{\lambda}{4} \qquad \ldots (20)$$

These points where displacement amplitude is zero, are called displacement nodes. The distance between two successive nodes is also $\lambda / 2$.

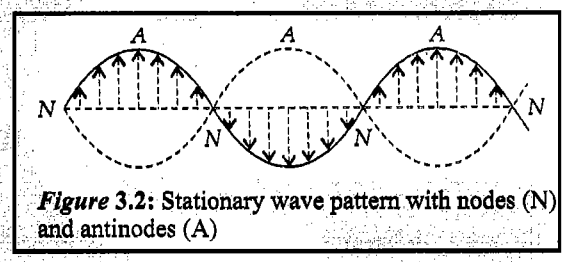


***Figure* 3.2:** Stationary wave pattern with nodes (N) and antinodes (A)

The distance between a node and its nearest antinode is $\lambda / 4$. The portion of the stationary wave pattern between two consecutive nodes is called a loop. All the particles within a loop vibrate with same phase but with different amplitudes (Fig. 3.2). The particles in two consecutive loops vibrate in opposite phase.

Ex. 15. Consider a pipe of length 65 cm with both ends open. How many modes of oscillations of the air column are possible up to a frequency of 1250 Hz ? (Take the speed of sound to be 330 m/s)

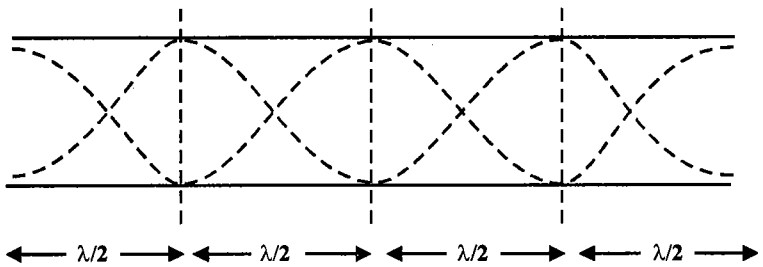
(a) 2 (b) 3 (c) 5 (d) 6

[JNU : 2010]

Soln. (c)

$$\therefore \; p\frac{\lambda}{2} = \ell \qquad \Rightarrow p\frac{v}{2n} = \ell \qquad \Rightarrow p = \frac{2n\ell}{v} = \frac{(1250)\times 2(0.65)}{330} = 4.92 \simeq 5$$

where 'p' is the number of modes of vibration 'v' is speed of sound, n is the frequency.



Ex. 16. A wave represented by the equation $y = a\cos(kx - \omega t)$ is superimposed with another wave to form a stationary wave such that point $x = 0$ is a node. The equation for the other wave is

(a) $a\sin(kx + \omega t)$ (b) $-a\cos(kx - \omega t)$ (c) $-a\cos(kx + \omega t)$ (d) $-a\sin(kx - \omega t)$

[IIT-JEE : 1988]

Soln. (a)

For a stationary wave to form, two identical waves should travel in opposite direction. Further at $x = 0$, resultant y (from both the waves) should be zero at all instants.

Ex. 17. A vibrating string of certain length l under a tension T resonates with a mode corresponding to the first overtone (third harmonic) of an air column of length 75 cm inside a tube closed at one end. The string also generates 4 beats/s when excited along with a tuning fork of frequency n. Now when the tension of the string is slightly increased the number of beats reduces to 2 per second. Assuming the velocity of sound in air to be 340 m/s, the frequency n of the tuning fork in Hz is

(a) 344 (b) 336 (c) 117.3 (d) 109.3

[IIT-JEE : 2008]

Soln. (a)

With increase in tension, frequency of vibrating string will increase. Since number of beats are decreasing. Therefore, frequency of vibrating string or third harmonic frequency of closed pipe should be less than the frequency of tuning fork by 4.

$\therefore$ Frequency of tuning fork

= third harmonic frequency of closed pipe + 4

$$= 3\left(\frac{v}{4l}\right) + 4 = 3\left(\frac{340}{4 \times 0.75}\right) + 4 = 344 \text{ Hz}$$

3.3.3 Energy density in stationary waves:

In a stationary wave formed by two identical waves propagating in two opposite directions the displacement at any point x at any time t is given by

$$\xi(x,t) = 2a\cos kx\cos\omega t \qquad \ldots (21)$$

Let us consider the stationary waves formed within a tube of the medium of length δx and area of cross-section unity. The total energy of an element of volume δx is given by the sum of K.E. and P.E.

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Thus, $$\delta E = \delta E_k + \delta E_p = \frac{1}{2}\rho_0 \delta x\left(\frac{\partial \xi}{dt}\right)^2 + \frac{1}{2}B\left(\frac{\partial \xi}{\partial x}\right)^2 \delta x$$

$$= \frac{1}{2}\rho_0 . 4a^2\omega^2 \cos^2 kx \sin^2 \omega t . \delta x + \frac{1}{2}B . 4a^2k^2 \sin^2 kx \cos^2 \omega t . \delta x$$

$$= 2\rho_0 a^2\omega^2\left[\cos^2 kx . \sin^2 \omega t + \sin^2 kx . \cos^2 \omega t\right]\delta x \qquad \ldots (22)$$

$$\overline{\delta E} = \rho_0 a^2\omega^2 \delta x \qquad \ldots (23)$$

Therefore, Average energy per unit volume, $\overline{E} = \rho_0 a^2\omega^2$ ... (24)

Thus energy density in a stationary wave is twice that of a progressive wave.

3.3.4 Energy flow in stationary waves:

Particle velocity, $u = \partial\xi / \partial t = -2a\omega\cos kx \sin\omega t$ ... (25)

and acoustic pressure, $p = -B(\partial\xi / \partial x) = 2aBk \sin kx \cos\omega t$ ... (26)

Now the rate of work done per unit area of the wavefront i.e., the rate of energy flow through unit area is

given by force per unit area × displacement in unit time = p × u

$$= -4a^2B\omega k \sin kx \cos wt . \cos kx \sin \omega t$$

$$= -a^2B\omega k \sin 2kx . \sin 2\omega t \quad \ldots (27)$$

As the time average value of sin 2ωt is zero, there is no transmission of energy through any cross-section in an ideal stationary wave.

Ex. 18. A string under tension T can support transverse waves with speed c. A displacement pattern of the form $y(x, t) = A \cos(2\pi x/\lambda) \cos(2\pi ct/\lambda)$ is set up in the string. The energy transported in a time interval $\Delta t = \lambda/c$ across $x = 0$ is

(a) 0 (b) TA (c) $T\lambda$ (d) $\frac{TA^2}{2\lambda}$ **[IISc : 2012]**

Ans. (a) $y(x,t) = A\cos\left(\frac{2\pi}{\lambda}x\right)\cos\left(2\pi\frac{c}{\lambda}t\right)$ or $y(x,t) = A\cos kx \cos \omega t$, represents standing wave. Energy transmission in time interval $\Delta t = \frac{\lambda}{c} = T$ (Time period) through only cross sectional area is zero.

Ex. 19. In a standing wave, the phase difference between any two points confined between adjacent node and antinode **[DU 2015]**

(a) is always zero (b) is between $\pi/4$ and $\pi/2$

(c) is between $\pi/2$ and π (d) changes periodically

Soln. **(a)** is always zero

3.4 DIFFERENCE BETWEEN PROGRESSIVE AND STATIONARY WAVES:

(1) A progressive wave originates from the regular vibration of a source. A stationary wave is formed by the superposition of two oppositely propagating waves.

(2) In progressive wave each particle oscillates with the same frequency and the same amplitude. In a stationary wave particles vibrate with the same frequency but with different amplitudes; at nodes amplitude is zero and at antinodes it is maximum.

(3) In progressive wave phase difference between two particles is proportional to the separation. In stationary waves all the particles within a loop vibrate in same phase and in the next loop the phase is opposite.

(4) In progressive wave variation of acoustic pressure is same at all points, but in a stationary wave variation of acoustic pressure is maximum at displacement nodes and zero at displacement antinodes.

(5) Progressive waves can carry energy from one place to another but in stationary waves there is no transfer of energy.

SOLVED EXAMPLES

1. Show that the function $\psi(x,t) = f_1(ct - x) + f_2(ct + x)$ satisfies the one dimensional differential wave equation.

Soln. $\frac{\partial\psi}{\partial x} = -f_1'(ct - x) + f_2'(ct + x)$

$$\frac{\partial^2\psi}{\partial x^2} = f_1''(ct - x) + f_2''(ct + x) \qquad \ldots (1)$$

where $f'(ct \pm x)$ indicate differentiation with respect the argument.

Again $\frac{\partial\psi}{\partial t} = cf_1'(ct - x) + cf_2'(ct + x)$

$$\frac{\partial^2\psi}{\partial t^2} = c^2\left[f_1''(ct - x) + f_2''(ct + x)\right] \qquad \ldots (2)$$

combining (1) and (2) we get $\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}$

Thus ψ satisfies the differential wave equation.

2. Compare the intensities of sound in air and water at N.T.P for the same acoustic pressure. Given the density of air $\rho_a = 1.21\,kg/m^3$, density of water $\rho_\omega = 998\,kg/m^3$, the speed of sound in air $c_a = 343$ m/s and the speed of sound in water $c_\omega = 1480$ m/s.

Soln. Intensity $I = p_{r.m.s}^2/\rho c$

Therefore, $\frac{I_{air}}{I_{water}} = \frac{p_{r.m.s}^2/(\rho c)_{air}}{p_{r.m.s}^2/(\rho c)_{water}} = \frac{\rho_\omega}{\rho_a} \times \frac{c_\omega}{c_a} = \frac{998}{1.21} \times \frac{1480}{343} = 3559$

3. Find the sound energy density in air of a free plane progressive acoustic wave having an IL of 80 dB. Given the velocity of sound in air = 343 m/s and reference intensity $I_0 = 10^{-12}\,W/m^2$.

Soln. $IL = 10\log_{10}\frac{I}{I_0} = 80\,dB$ or, $I = 10^8\,I_0 = 10^8 \times 10^{-12}\,W/m^2 = 10^{-4}\,W/m^2$

Now energy density $= \frac{E}{V} = \frac{E}{tAc} = \frac{I}{c} = \frac{10^{-4}}{343}\,J/m^3 = 2.9\times 10^{-7}\,J/m^3$

4. An intense plane harmonic wave in air STP has an r.m.s excess pressure of $20\,N/m^2$. The frequency is 1 kHz and the phase velocity 350 m/s. Calculate the maximum value of particle displacement and particle velocity. Given $\gamma_{air} = 1.40$.

Soln. Let displacement $\xi = a\cos(\omega t - kx)$

Excess pressure $p = -B\frac{\partial\xi}{\partial x} = Bak\sin(\omega t - kx)$

where bulk mudulus $B = \gamma.P_0 = 1.40 \times 1.013 \times 10^5\,N/m^2$.

$p_{max} = Bak$ or, Maximum particle displacement

$$a = \frac{p_{max}}{Bk} = \frac{\sqrt{2}p_{r.m.s} \times c}{\gamma P_0 \times \omega} \qquad [\because c = \omega/k] = \frac{\sqrt{2} \times 20 \times 350}{1.40 \times 1.013 \times 10^5 \times 2\pi \times 10^3}\,m = 1.11 \times 10^{-5}\,m$$

Particle velocity $u = \partial\xi/\partial t = -a\omega\sin(\omega t - kx) = a\omega\cos(\omega t - kx + 90^\circ)$

Therefore, maximum particle velocity $= a\omega = 1.11 \times 10^{-5} \times 2\pi \times 10^3\,m/s = 6.97 \times 10^{-2}\,m/s$.

5. Write the wave function for a plane harmonic progressive wave which is progressing with velocity 340 m/s along a direction having direction cosines $1/\sqrt{2}$, $1/\sqrt{3}$ and $1/\sqrt{6}$

Soln. $\psi(x,y,z,t) = a\cos\frac{2\pi}{\lambda}(ct - lx - my - nz) = a\cos\frac{2\pi}{\lambda}\left(340t - \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{6}}z\right)$

6. Two plane progressive harmonic waves propagating along the same line but moving in opposite direction overlap. They are represented by

$$y_1 = a_1 \sin(\omega t - kx) \text{ and } y_2 = a_2 \sin(\omega t + kx), \qquad a_1 > a_2$$

Show that the resultant wave consists of a stationary wave together with a progressive wave moving along the x-axis.

Soln. The resultant wave is given by

$$y = y_1 + y_2 = a_1 \sin(\omega t - kx) + a_2 \sin(\omega t + kx)$$

Let $a_1 = a_2 + b$, where b is a positive quantity.

Therefore, $y = 2a_2 \cos kx \sin \omega t + b \sin(\omega t - kx)$

Obviously the first term represents a stationary wave and the second term represents a progressive wave moving along positive x-direction.

7. A long string having a cross-sectional area 0.80 mm² and density 12.5 g/cm³, is subjected to a tension of 64 N along the X-axis. One end of this string is attached to a vibrator moving in transverse direction at a frequency of 20 Hz. At $t = 0$, the source is at a maximum displacement $y = 1.0$ cm

(a) Find the speed of the wave travelling on the string

(b) Write the equation for the wave

(c) What is the displacement of the particle of the string at $x = 50$ cm at time $t = 0.05$ s ?

(d) What is the velocity of this particle at this instant ?

Soln. (a) The mass of 1 m long part of the string is

$$m = (0.80 \text{ mm}^2) \times (1\text{m}) \times (12.5 \text{ g/cm}^3)$$
$$= (0.80 \times 10^{-6} \text{ m}^3) \times (12.5 \times 10^3 \text{ kg/m}^3)$$
$$= 0.01 \text{ kg.}$$

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The linear mass density is $\mu = 0.01$ kg/m . The wave speed is

$$v = \sqrt{F/\mu}$$
$$= \sqrt{\frac{64 \text{ N}}{0.01 \text{ kg/m}}} = 80 \text{ m/s}$$

(b) The amplitude of the source is $A = 1.0$ cm and the frequency is $v = 20$ Hz. The angular frequency is $\omega = 2\pi v = 40\pi s^{-1}$. Also at $t = 0$, the displacement is equal to its amplitude i.e., at $t = 0$, $x = A$. The equation of motion of the source is, therefore,

$$y = (1.0 \text{ cm}) \cos[(40\pi s^{-1})t] \qquad \ldots \text{(i)}$$

The equation of the wave travelling on the string along the positive X-axis is obtained by replacing t with $t - x/v$ in equation (i). It is, therefore,

$$y = (1.0 \text{ cm}) \cos\left[(40\pi s^{-1})\left(t - \frac{x}{v}\right)\right]$$

$$= (1.0\,\text{cm})\cos\left[(40\pi s^{-1})t - \left(\frac{\pi}{2}m^{-1}\right)x\right], \qquad \text{... (ii)}$$

where the value of v has been put from part (a).

(c) The displacement of the particle at $x = 50$ cm at time $t = 0.05\,s$ is by equation (ii),

$$y = (1.0\,\text{cm})\cos\left[(40\pi s^{-1})(0.05\,s) - \left(\frac{\pi}{2}\text{m}^{-1}\right)(0.5\,\text{m})\right]$$

$$= (1.0\,\text{cm})\cos\left[2\pi - \frac{\pi}{4}\right] = \frac{1.0\,\text{cm}}{\sqrt{2}} = 0.71\,\text{cm}$$

(d) The velocity of the particle at position x at time t is, by equation (ii),

$$v = \frac{\partial y}{\partial t} = -(1.0\text{ cm})(40\pi s^{-1})\sin\left[(40\pi s^{-1})t - \left(\frac{\pi}{2}\text{m}^{-1}\right)x\right].$$

Putting the values of x and t,

$$v = -(40\pi\text{ cm/s})\sin\left(2\pi - \frac{\pi}{4}\right) = \frac{40\pi}{\sqrt{2}}\text{cm/s} \approx 89\text{ cm/s}.$$

3. The vibrations of a string fixed at both ends are described by the equation

$$y = (5.00\text{ mm})\sin[(1.57\,\text{cm}^{-1})x]\sin[(314\,s^{-1})t]$$

(a) What is the maximum displacement of the particle at $x = 5.66$ cm ?

(b) What are the wavelengths and the wave speeds of the two transverse waves that combine to give the above vibration ?

(c) What is the velocity of the particle at $x = 5.66$ cm at time $t = 2.00\,s$?

(d) If the length of the string is 10.0 cm, locate the nodes and the antinodes. How many loops are formed in the vibration ?

Soln. (a) The amplitude of the vibration of the particle at position x is $A = \left|(5.00\text{ mm})\sin[(1.57\text{ cm}^{-1})x]\right|$.

For $x = 5.66$ cm,

CAREER ENDEAVOUR

$$A = \left|(5.00\text{ mm})\sin\left[\frac{\pi}{2}\times 5.66\right]\right|$$

$$= \left|(5.00\text{ mm})\sin\left(2.5\pi + \frac{\pi}{3}\right)\right|$$

$$= \left|(5.00\text{ mm})\cos\frac{\pi}{3}\right| = 2.50\text{ mm}.$$

(b) From the given equation, the wave number $k = 1.57\text{ cm}^{-1}$ and the angular frequency $\omega = 314\,s^{-1}$.

Thus, the wavelength is

$$\lambda = \frac{2\pi}{k} = \frac{2\times 3.14}{1.57\text{ cm}^{-1}} = 4.00\text{ cm}$$

and the frequency is $\nu = \dfrac{\omega}{2\pi} = \dfrac{314\,s^{-1}}{2\times 3.14} = 50\,s^{-1}$.

The wave speed is $v = \nu\lambda = (50\,s^{-1})(4.00\,\text{cm}) = 2.00$ m/s.

(c) The velocity of the particle at position x at time t is given by

$$v = \frac{\partial y}{\partial t} = (5.00 \text{ mm}) \sin[(1.57 \text{ cm}^{-1})x], [314 s^{-1} \cos(314 s^{-1})t]$$

$$= (157 \text{ cm/s}) \sin(1.57 \text{ cm}^{-1})\, x \cos(314 s^{-1})t.$$

Putting $x = 5.66$ cm and $t = 2.00 s$, the velocity of this particle at the given instant is

$$(157 \text{ cm/s}) \sin\left(\frac{5\pi}{2} + \frac{\pi}{3}\right) \cos(200\pi) = (157 \text{ cm/s}) \times \cos\frac{\pi}{3} \times 1 = 78.5 \text{ cm/s}.$$

(d) The nodes occur where the amplitude is zero i.e., $\sin(1.57 \text{ cm}^{-1})\, x = 0$.

or $\left(\frac{\pi}{2} \text{cm}^{-1}\right) x = n\pi$, where n is an integer.

Thus, $x = 2n$ cm.

The nodes, therefore, occur at $x = 0$, 2 cm, 4 cm, 6 cm, 8 cm and 10 cm. Antinodes occur in between them i.e., at $x = 1$ cm, 3 cm, 5 cm, 7 cm and 9 cm. The string vibrates in 5 loops.

Chapter 4

Phase And Group Velocity

4.1 WAVE TRAIN AND THE CONCEPT OF WAVE GROUP:

A wave train of finite length may be considered to be formed by the superposition of a theoretically infinite number of plane harmonic waves having continuously differing frequencies. For practical purpose the frequencies are limited within a finite range depending on the length of the wave train. The shorter is the length of the wave train the wider is the effective frequency range. In this sense a short enough wave train may be considered as a group of harmonic waves. It may be called a wave group or a wave packet.

4.2 PHASE VELOCITY, GROUP VELOCITY AND THEIR RELATIONSHIP:

When a monochromatic harmonic wave moves through a medium the velocity with which planes of constant phase move is called the phase velocity or wave velocity. A plane harmonic wave may be represented by $\psi(x,t) = \text{Re } Ae^{j(\omega t - kx)}$ ... (1)

For planes of constant phase, $\omega t - kx = \text{constant}$ ... (2)

Differentiating, $\frac{d}{dt}(\omega t - kx) = 0$ or, $\frac{dx}{dt} = \frac{\omega}{k} = c_p$ ... (3)

where c_p is called the phase velocity.

Practical waves are of finite duration and are not truly monochromatic. A short enough wave train may be considered as a wave group formed by the superposition of an infinite number of plane harmonic wave having slightly differing frequencies and phases. The wave group has a maximum of amplitude which falls off to zero not far from the maximum. If the medium is dispersive i.e., the phase velocity depends on frequency then the shape of the wave group changes as it travels through the medium. The maximum of the wave group travels with a velocity that is different from the velocities of the component waves. This velocity is known as group velocity (c_g).

4.3 RELATION BETWEEN PHASE VELOCITY AND GROUP VELOCITY:

The wave group is not situated at a fixed point but moves in space with a velocity, called group velocity, given by $c_g = \frac{dx}{dt} = \frac{d\omega}{dk}$ ... (4)

Since phase velocity $c_p = \omega / k \Rightarrow \omega = c_p k$ we can write

$$c_g = \frac{d}{dk}(c_p k) = c_p + k\frac{dc_p}{dk} = c_p + k\frac{dc_p}{d\lambda}\cdot\frac{d\lambda}{dk}$$

Now, $k = 2\pi/\lambda$ or, $d\lambda/dk = -2\pi/k^2$.

$$\Rightarrow \qquad c_g = c_p - \frac{2\pi}{k}.\frac{dc_p}{d\lambda} = c - \lambda\frac{dc_p}{d\lambda}$$

This is the required relation between phase velocity and group velocity for a dispersive medium. For a non-dispersive medium $dc_p/d\lambda = 0$. Then $v_g = c_p$ i.e, group velocity equals phase velocity.

SOLVED EXAMPLES

1. For gravity waves in a liquid the phase velocity c depends on the wavelength λ according to the formula $c_p = A\sqrt{\lambda}$, A being a constant. Show that the group velocity is half the phase velocity.

Soln. $c_p = A\sqrt{\lambda} \quad \Rightarrow \frac{dc_p}{d\lambda} = \frac{1}{2}A\lambda^{-1/2}$

Therefore, $c_g = c_p - \lambda\frac{dc_p}{d\lambda} = c_p - \frac{1}{2}A\sqrt{\lambda} = c_p - \frac{c_p}{2} = \frac{1}{2}c_p$

2. The dispersion relation for microwaves in ionosphere is given by

$$\omega^2 = \omega_p^2 + c^2k^2$$

where c is the velocity of light in free space and ω_p is a constant depending on the electron density of the ionosphere. Show that the phase velocity c_p is greater than c. Does it violate the principle of relativity?

Soln. $\omega = \sqrt{\omega_p^2 + c^2k^2}$

Therefore, Phase velocity $c_p = \frac{\omega}{k} = \sqrt{c^2 + \frac{\omega_p^2}{k^2}}$ which is greater than c.

Differentiating the dispersion relation with respect to k we get

$$2\omega\frac{d\omega}{dk} = c^2.2k \Rightarrow \frac{\omega}{k}.\frac{d\omega}{dk} = c^2$$

$$\Rightarrow \qquad c_pc_g = c^2 \qquad [\text{since } c_p > c,\ c_g < c]$$

It does not violate the principle of relativity because energy or signal is transmitted not with the phase velocity but with group velocity which is less than c.

3. Two harmonic waves represented by

$$\xi_1 = 3\cos(7t - 10x)\,m \text{ and } \xi_2 = 3\cos(5t - 8x)\,m$$

are superposed to form a wave group. Find the group velocity.

Soln. Angular frequencies are $\omega_1 = 7$ rad / s and 5 rad/s. Wave numbers are $k_1 = 10\,m^{-1}$ and $k_2 = 8\,m^{-1}$.

Therefore, Group velocity $c_g = \frac{\Delta\omega}{\Delta k} = \frac{7-5}{10-8} = 1\,m/s$.

4. The phase velocity c_p of surface waves on a liquid of density ρ and surface tension S is given by

$$c_p^2 = \frac{g\lambda}{2\lambda} + \frac{2\pi S}{\rho\lambda}$$

where 'g' is the accleration due to gravity and λ is the wavelength of the wave.

(i) Obtain an expression for the group velocity of the wave.

(ii) Calculate the wavelength (λ_0) for which c is the minimum

(iii) Show that for $\lambda = \lambda_0$, group velocity $c_g = c_p$.

oln. (i) Group velocity $c_g = \frac{d\omega}{dk} = \frac{d}{dk}(c_p k) = c_p + k\frac{dc_p}{dk}$

Since $k = \frac{2\pi}{\lambda}$, $c_g = c_p - \lambda\frac{dc_p}{d\lambda}$

Now differentiating the given expression with respect to λ,

$$2c_p\frac{dc_p}{d\lambda} = \frac{g}{2\pi} - \frac{2\pi S}{\rho\lambda^2}$$

$$\Rightarrow \quad c_g = c_p - \frac{\lambda}{2c_p}\left(\frac{g}{2\pi} - \frac{2\pi S}{\rho\lambda^2}\right)$$

(ii) For c to be minimum, $\frac{dc}{d\lambda} = 0 \quad \Rightarrow \frac{g}{2\pi} - \frac{2\pi S}{\rho\lambda_0^2} = 0 \quad \Rightarrow \lambda_0 = 2\pi\sqrt{S/\rho g}$

(iii) For $\lambda = \lambda_0 \quad \Rightarrow c_g = c_p - \frac{\lambda_0}{2c_p}\left(\frac{g}{2\pi} - \frac{2\pi S}{\rho.4\pi^2(S/\rho g)}\right) = c_p - 0 = c_p$

A wave packet in a certain medium is constructed by superposing waves of frequency ω around $\omega_0 = 100$ and the corresponding wave-number k with $k_0 = 10$ as given in the table below

ω	k
81.00	9.0
90.25	9.5
100.00	10.0
110.25	10.5
121.00	11.0

Find the ratio v_g / g_p of the group velocity v_g and the phase velocity v_p.

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{3}{2}$ (d) 2

[IIT-JAM : 2009]

)ln. **(d)** $v_p = \frac{\omega}{k} = \frac{100}{10} = 10$; $v_g^{(1)} = \frac{100-90.25}{10-9.5} = \frac{9.75}{0.5} = 19.5$

$v_g^{(2)} = \frac{110.25-100}{10.5-10} = \frac{10.25}{0.5} = 20.5$; $v_g = \frac{v_g^{(1)} + v_g^{(2)}}{2} = \frac{19.5+20.5}{2} = 20$

$\therefore \frac{v_g}{v_p} = \frac{20}{10} = 2$

6. For a wave in a medium the angular frequency 'ω' and the wave vector $\vec{k}$ are related by, $\omega^2 = \left(\omega_0^2 + c^2k^2\right)$, where ω_0 and c are constants. The product of group and phase velocities, i.e. $v_g v_p$ is:

(a) $0.25c^2$ (b) $0.4c^2$ (c) $0.5c^2$ (d) c^2

[IIT-JAM : 2010]

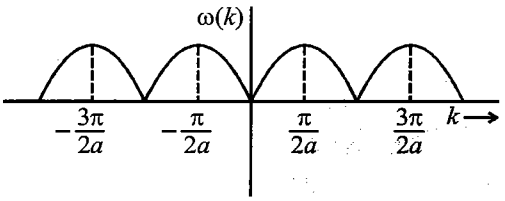
Soln. **(d)** $v_p = \dfrac{\omega}{k} = \dfrac{\sqrt{\omega_0^2 + c^2k^2}}{k}$; $v_g = \dfrac{d\omega}{dk} = \dfrac{c^2k}{\sqrt{\omega_0^2 + c^2k^2}}$ $\Rightarrow \boxed{v_g.v_p = c^2}$

7. The dispersion relation for a wave is given as $\omega(k) = C|\sin ka|$, where C is a constant. The group velocity of the wave vanishes at

(a) $k = 0$ (b) $k = \pm \pi/2a$ (c) $k = \pm \pi/2$ (d) $k = \pm 2\pi/a$

[IIT-JAM GP : 2008]

Soln. **(b)**



$\omega(k) = C|\sin ka|$

$v_g = \dfrac{d\omega}{dk} = 0$ at $k = \pm\dfrac{\pi}{2a}$

8. In a certain medium, the wave number k and the frequency ω are related by the dispersion relation $\omega^2 = c^2k^2(1 + \alpha k^2)$, where c and α are constants. If v_g is the group velocity and v_p is the phase velocity, then

(a) $v_g = v_p$ (b) $\dfrac{v_g}{v_p} = \dfrac{(1+\alpha k^2)}{(1+2\alpha k^2)}$ (c) $v_g \cdot v_p = c^2$ (d) $v_g \cdot v_p = c^2(1 + 2\alpha k^2)$

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[IIT-JAM GP : 2010]

Soln. **(d)**

$\omega = ck(1 + \alpha k^2)^{1/2}$

$v_g = \dfrac{d\omega}{dk} = c(1 + \alpha k^2)^{1/2} + ck\dfrac{1}{2}(1 + \alpha k^2)^{1/2}(2\alpha k)$

$= c(1 + \alpha k^2)^{1/2} + \dfrac{\alpha ck^2}{(1 + \alpha k^2)^{1/2}} = \dfrac{c(1 + \alpha k^2) + \alpha ck^2}{(1 + \alpha k^2)^{1/2}}$

$= \dfrac{c(1 + 2\alpha k^2)}{(1 + \alpha k^2)^{1/2}}$

Since $v_p = \dfrac{\omega}{k} = c(1 + \alpha k^2)^{1/2}$

Therefore, $v_g \cdot v_p = c^2(1 + 2\alpha k^2)$

9. The dispersion relation for surface waves propagating in a fluid is given as: $\omega^2 = \alpha k + \beta k^3$, where α and β are constants with appropriate units. The phase velocity βV_p becomes equal to the group velocity V_g at $k = k_0$. The value of k_0 is

(a) $\sqrt{\frac{\alpha}{2\beta}}$ (b) $\sqrt{\frac{2\alpha}{\beta}}$ (c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\sqrt{\frac{3\alpha}{\beta}}$

[IIT-JAM GP: 2012]

Soln. **(c)** $\omega = \sqrt{\alpha k + \beta k^3}$

Phase velocity $V_p = \frac{\omega}{k} = \frac{\sqrt{\alpha k + \beta k^3}}{k}$

Group velocity $V_y = \frac{d\omega}{dk} = \frac{(\alpha + 3\beta k^2)}{2\sqrt{\alpha k + \beta k^3}}$

$V_p = V_g$

$\Rightarrow \frac{\sqrt{\alpha k + \beta k^3}}{k} = \frac{\alpha + 3\beta k^2}{2\sqrt{\alpha k + \beta k^3}} \Rightarrow 2(\alpha k + \beta k^3) = \alpha k + 3\beta k^3$

$\Rightarrow \beta k^3 - \alpha k = 0$

$k = \sqrt{\frac{\alpha}{\beta}}$

10. The angular frequency ω of deep water waves varies as the inverse square root of the wavelength λ, *i.e.*, $\omega(\lambda) \propto 1/\sqrt{\lambda}$. Which of the following is the relation between its group velocity v_g and phase velocity v_p?

(a) $v_g = v_p/2$ (b) $v_g = v_p$ (c) $v_g = 2v_p$ (d) $v_g = \lambda v_p$

[JNU : 2009]

Soln. **(a)** $\omega \propto \frac{1}{\sqrt{\lambda}}$; $\omega = \frac{a}{\sqrt{\lambda}}$, '$a$' is constant

$\therefore \quad \omega = \frac{a}{\sqrt{2\pi}}.\sqrt{k} \qquad \because \lambda = \frac{2\pi}{k}$

Or, $\omega^2 = \frac{a^2}{2\pi}k$

$\therefore \quad 2\omega\frac{d\omega}{dk} = \frac{a^2}{2\pi}; \qquad \frac{d\omega}{dk} = \frac{a^2}{4\pi}.\frac{1}{\omega} = \frac{a^2}{4\pi}.\frac{\sqrt{2\pi}}{a\sqrt{k}}$

$\Rightarrow \quad \frac{d\omega}{dk} = \frac{a}{2\sqrt{2}\pi}.\frac{1}{\sqrt{k}} = v_g$ and $\frac{\omega}{k} = \frac{a}{\sqrt{2\pi}}.\frac{\sqrt{k}}{k} = \frac{a}{\sqrt{2\pi}}.\frac{1}{\sqrt{k}} = v_p$

$\Rightarrow \quad v_g = \frac{1}{2}\left(\frac{a}{\sqrt{2\pi}}.\frac{1}{k}\right) = \frac{1}{2}v_p \Rightarrow \boxed{v_g = \frac{v_p}{2}}$

Chapter 5

Doppler Effect

5.1 INTRODUCTION:

The pitch of the note emitted by a source of sound appears to change when there is a relative motion between the source and the listener with respect to each other or to the medium. This apparent change in the pitch of a note is known as Doppler effect in sound. It is our common experience that the pitch or frequency of whistling railway engine appears to increase as it approaches a listener and the pitch or frequency suddenly drops as it passes and recedes from the listener. In case of a moving source the change in frequency can be explained as due to the fact that, though the source is emitting waves at constant rate, the waves at constant rate, the waves emitted in a given time will now occupy a shorter or longer distance than when the source is stationary. Consequently the apparent wavelength and hence the frequency appears to change. In case a listener is moving towards or away from a source, the wavelength remain unchanged but the listener encounters a larger or smaller number of waves in a given time than if he or she remained stationary. The change in frequency due to the motion of the medium is associated with the change in the velocity of sound. The fundamental fact underlying the Doppler effect is that Doppler shift in frequency occurs when the transit time from the source to the observer is different for different wavefronts.

5.2 CALCULATION OF DOPPLER SHIFT:

5.2.1 Source moving towards a stationary observer:

Let in Figure-5.1 S be source moving with a velocity υ_s towards the stationary observer O. The medium is assumed to be at rest. If ν is the frequency of the source, it will emit ν complete waves in 1 sec. If the source is at rest these ν waves will occupy a length SP $(=\lambda) = c$, where c is speed of wave. However, if the source

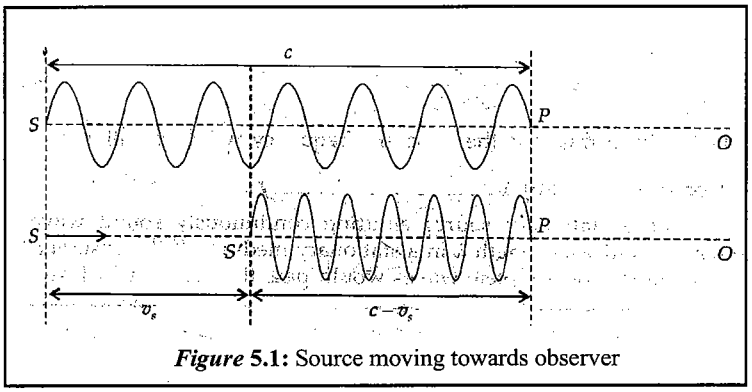


Figure **5.1:** Source moving towards observer

moves with velocity υ_s, in one second, it goes from S to S' where SS' = υ_s. As the velocity of sound is independent of the motion of the source, the ν waves emitted during this one second will now occupy a smaller length $c - \upsilon_s = S'P$.

The original wavelength $\lambda = c/\nu$ will now be changed to $\lambda' = \frac{c - \upsilon_s}{\nu}$. The observer will interpret them

as waves of frequency $\nu' = \frac{c}{\lambda'} = \frac{c}{c - \upsilon_s}\nu$ ... (1)

Obviously $\nu' > \nu$. As ν_s is increased from zero, ν' increases. At $\upsilon_s = c$, ν' becomes infinite. In this case waves travel with the source and reach the observer at the same time.

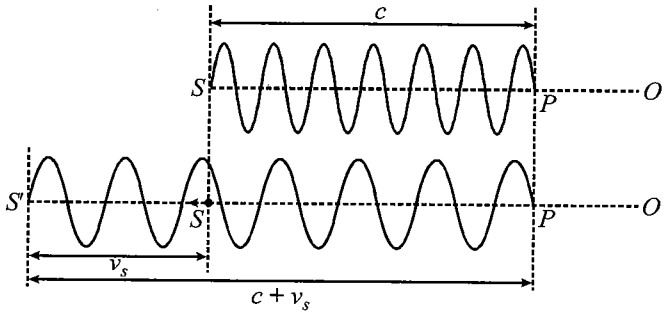


***Figure* 5.2 :** Source moving away from the observer

If $\upsilon_s > c$ then ν' is negative which means that the waves emitted by the source are received in the reversed order. In this case the source moves ahead and waves follow it.

Source moving away from station are observer:

If the source moves away from the observer the ν waves emitted in 1 sec. will now occupy a greater length $c + \upsilon_s$ as shown in Figure-5.2. So the new wavelength in this case would be $\lambda' = \frac{c + \upsilon_s}{\nu}$ and hence the apparent frequency would appear to be

$$\nu' = \frac{c}{\lambda'} = \frac{c}{c + \upsilon_s}.\nu \qquad ...(2)$$

Obviously in this case the apparent frequency will be smaller.

Ex. 1. The engine of a train, emitting the sound of frequency ν_0 approaches an observer with constant speed. If the observer measures the frequencies as ν_1 when it is approaching and ν_2 while it is going away, the relation between the frequencies is given by

(a) $\nu_1 = \nu_2 = \nu_0$ (b) $\nu_1 > \nu_0 > \nu_2$ (c) $\nu_1 < \nu_0 < \nu_2$ (d) $\nu_1 = \nu_2 \neq \nu_0$

[IIT-JAM GP : 2008]

Soln. **(b)** When source is moving the towards the observer, the apparent frequency is more then the actual frequency and vice-versa.

Ex. 2. A siren of natural frequency 600 Hz is fitted on an ambulance. When it approaches a pedestrian, she heard a frequency of 640 Hz. At what speed was the ambulance approaching her? (Note that the speed of sound under those conditions was 360 m/s)

(a) 24.0 km/hr (b) 30.2 km/hr (c) 60.8 km/hr (d) 86.4 km/hr

[IISc : 2007]

Soln. **(a)**

When source (ambulance) is moving towards stationary observer, aparent frequency is given by

$$\nu' = \nu_0\left(\frac{c}{c - v_s}\right)$$

$$\Rightarrow \quad 640 = 600\left(\frac{360}{360 - v_s}\right)$$

$v_s = 20.7$ km/hr

Ex. 3. A train moves towards a stationary observer with speed 34 m/s. The train sounds a whistle and its frequency registered by the observer is f_1. If the train's speed is reduced to 17 m/s, the frequency registered is f_2. If the speed of sound is 340 m/s then the ratio f_1/f_2 is

(a) 18/19 (b) 1/2 (c) 2 (d) 19/18

[IIT-JEE : 2000]

Soln. **(d)**

$$f_1 = \left(\frac{c}{c - v_s}\right) f_0$$

$$f_1 = f\left(\frac{340}{340 - 34}\right) = f\left(\frac{340}{306}\right) \text{ and } f_2 = f\left(\frac{340}{340 - 17}\right) = f\left(\frac{340}{323}\right)$$

$$\therefore \quad \frac{f_1}{f_2} = \frac{323}{306} = \frac{19}{18}$$

5.2.2 Observer moving towards a stationary source:

Let S be a stationary source emitting continuously sound waves of frequency ν and wavelength λ in a stationary medium. If the observer O is at rest, ν number of complete waves would pass the observer in 1 sec. and

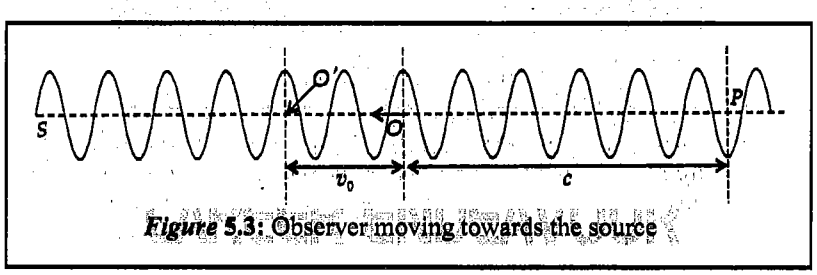


Figure 5.3: Observer moving towards the source

occupy the length OP = c, where c is speed of sound. However, if the observer moves with velocity υ_0, in 1 sec. it moves from O to O' where $OO' = \upsilon_0$. So all the waves contained in the length $O'P = c + \upsilon_0$ will be received by the observer in 1 sec. Here the wavelength remain unchanged hence the number of waves in the length $O'P$ i.e., the apparent frequency

$$\nu' = \frac{c + \upsilon_0}{\lambda} = \frac{c + \upsilon_0}{c}.\nu \qquad \ldots (3)$$

Obviously $\nu' > \nu$ i.e., the frequency appears to increase when the observer moves towards the source S. Here wavelength remains unchanged.

The Doppler effect is due to the fact that the observer encounters a larger number of waves in a given time than when he or she was at rest relative to the source.

Observer moving away from the source:

If the observer moves away from the source (Figure-5.4) then only the waves contained in the length $c - v_0$ will be received by the listener in 1 sec. Therefore, apparent frequency in this case would be

$$\nu' = \frac{c - \upsilon_0}{c}\nu \qquad ...(4)$$

Obviously the frequency appears to decrease when the observer is moving away from the source. If $\upsilon_0 = c$ then $\nu' = 0$. If $\upsilon_0 = 2c$ then $\nu' = -\nu$, which indicates that the listener will hear the sound in correct time but backward. If $\upsilon_0 >> c, \nu' \to -\infty$; this means that sound waves lying ahead of the observer will be heard in reverse of the natural order but the waves

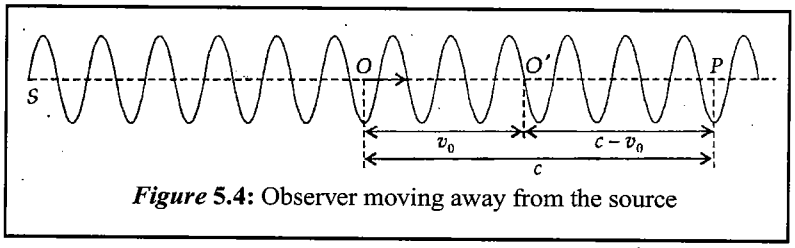


***Figure* 5.4:** Observer moving away from the source

lying behind the observer will never reach the observer i.e., he or she will hear nothing. Therefore, the above formulas are applicable when the speed of the source and observer is smaller than c.

x. 4. An object of specific gravity ρ is hung from a thin steel wire. The fundamental frequency for transverse standing waves in the wire is 300 Hz. The object is immersed in water, so that one half of its volume is submerged. The new fundamental frequency (in Hz) is

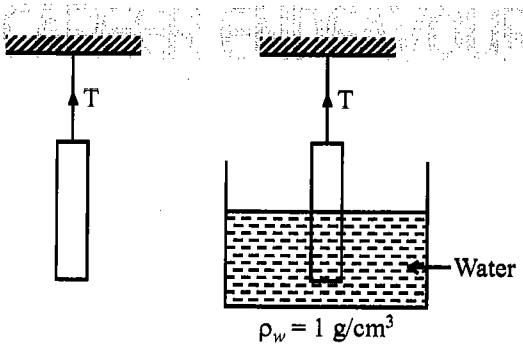
(a) $300\left(\frac{2\rho - 1}{2\rho}\right)^{1/2}$ (b) $300\left(\frac{2\rho}{2\rho - 1}\right)^{1/2}$ (c) $300\left(\frac{2\rho}{2\rho - 1}\right)$ (d) $300\left(\frac{2\rho - 1}{2\rho}\right)$

[IIT-JEE : 1995]

ln. (a)

The diagramatic representation of the given problem is shown in figure. The expression of fundamental frequency is

$$\nu = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$



In air $T = mg = (V\rho)g$

$$\therefore \quad \nu = \frac{1}{2l}\sqrt{\frac{V\rho g}{\mu}} \qquad ...(i)$$

When the object is half immersed in water

$$T' = mg - \text{upthrust} = V\rho g - \left(\frac{V}{2}\right)\rho_w g$$

$$= \left(\frac{V}{2}\right) g\,(2\rho - \rho_w)$$

The new fundamental frequency is

$$\nu' = \frac{1}{2l} \times \sqrt{\frac{T'}{\mu}} = \frac{1}{2l}\sqrt{\frac{(V/2)(2\rho - \rho_w)g}{\mu}} \qquad ...(ii)$$

$$\therefore \quad \frac{\nu'}{\nu} = \left(\sqrt{\frac{2\rho - \rho_w}{2\rho}}\right) \text{ or } \nu' = \nu\left(\frac{2\rho - \rho_w}{2\rho}\right)^{1/2} = 300\left(\frac{2\rho - 1}{2\rho}\right)^{1/2} \text{ Hz}$$

5.2.3 Both the source and observer are moving:

Suppose the source S and observer O both are moving along the same direction SO with speeds c and υ_0 respectively. Let the source S be emitting continuously sound waves of frequency ν and wavelength λ in a stationary medium. If the source were stationary the ν waves emitted in 1 sec. would

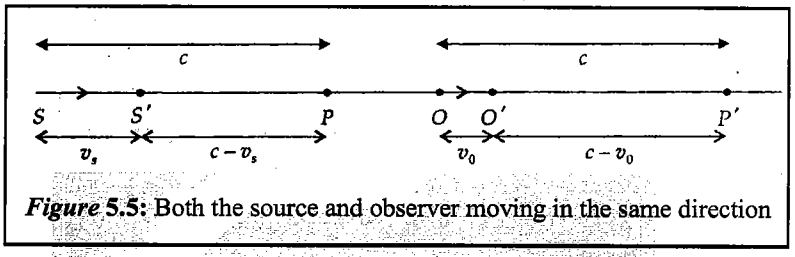


***Figure* 5.5:** Both the source and observer moving in the same direction

occupy the length SP = c, where c is the speed of sound. Now as the source moves through in 1 sec., the ν waves will now occupy the length S'P = $c - \upsilon_s$ and hence the original wavelength SS' (= λ) = υ_s will be changed to $\lambda' = \dfrac{c - \upsilon_s}{\nu}$

Now this wave of wavelength λ' is approaching the observer O. The number of waves received by a stationary observer is equal to the number of waves contained in OP' = c. As the observer moves in 1 sec. through OO' = υ_0, he or she will receive only those waves contained in the length O'P' = $c - \upsilon_0$. Therefore, the apparent frequency would be

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$$\nu' = \frac{c - \upsilon_0}{\lambda'} = \frac{c - \upsilon_0}{c - \upsilon_s} \cdot \upsilon \qquad ... (5)$$

This formula can also be obtained easily by combining the Equations (1) and (4).

If the observer moves towards the source then υ_0 is to be taken with a negative sign. Thus if the source and observer move towards each other then

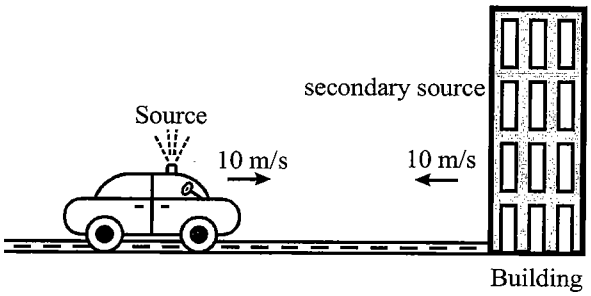
$$\nu' = \frac{c + \upsilon_0}{c - \upsilon_s}\nu \qquad ... (6)$$

Ex. 5. A police car with a siren of frequency 8 kHz is moving with uniform velocity 36 km/h towards a tall building which reflects the sound waves. The speed of sound in air is 320 m/s. The frequency of the siren heard by the car driver is

(a) 8.50 kHz (b) 8.25 kHz (c) 7.75 kHz (d) 7.50 kHz

[IIT-JEE : 2011]

Soln. (a)



In this case, wall will act like secondary source of sound moving towards car driver with speed equal to car.

Speed of observer (driver) $v_0 = -10\,\text{m/s}$

[minus sign implies that observer is moving towards secondary source (wall)]

Speed of the source $v_s = 10\,\text{m/s}$

Apparent frequency of sound heard by the driver will be

$$\nu' = \left(\frac{c-\upsilon_0}{c-v_s}\right)\nu = 8\left[\frac{320+10}{320-10}\right] = 8.5\ \text{kHz}$$

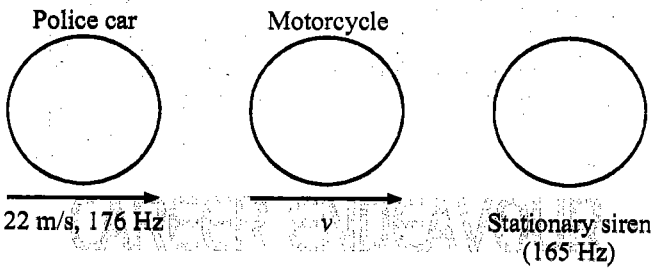
Ex. 6. A police car moving at 22 m/s chases a motorcyclist. The police man sounds his horn at 176 Hz, while both of them move towards a stationary siren of frequency 165 Hz. Calculate the speed of the motorcycle. If it is given that the motorcyclist does not observe any beats (speed of sound = 330 m/s)

(a) 33 m/s (b) 22 m/s (c) zero (d) 11 m/s

[IIT-JEE : 2003]

Soln. (b)

The motorcyclist observes no beats. So, the apparent frequency observed by him from the two sources must be equal.



$$f_1 = f_2$$

$$\therefore\ 176\left(\frac{330-v_o}{330-22}\right) = 165\left(\frac{330+v_o}{330}\right)$$

Solving this equation, we get $v_o = 22$ m/s.

5.2.4 Source, observer and medium all are moving

If the medium carrying the waves moves with a velocity υ_m with respect to some inertial frame of reference in the direction of sound, the apparent sound speed increase to $c + \upsilon_m$. In this in all the above formulas (1)–(6), c is to be replaced by $c + \upsilon_m$. On the other hand, if the medium moves against the direction of sound then the apparent sound speed becomes $c - \upsilon_m$ and we are to replace c by $c - \upsilon_m$ in each of the above formulas (1) – (6). Thus if the observer, source and the medium all are moving in the same direction then apparent frequency

$$\nu' = \frac{c + \upsilon_m - \upsilon_0}{c + \upsilon_m - \upsilon_s}.\nu \qquad \ldots (7)$$

Here ν is the frequency of the source, υ_s its speed towards the observer, υ_0 the speed of observer away from the source, υ_m the speed of the medium towards the observer and c the speed of sound in stationary medium.

If $\upsilon_0 = \upsilon_s$, then $\nu' = \nu$ i.e. when the velocities of the source and observer are the same in magnitude and direction, the velocity of the medium will have no effect on the frequency.

5.2.5 Source moving not in line with stationary observer:

In most practical cases the stationary observer O is situated away from the line of motion AB of the source (Figure-5.6).

When the source is at S_1 the component of the source velocity (υ_s) along S_1O is $\upsilon_s \cos\theta_1$ i.e., the source appears to move towards the observer with a velocity $\upsilon_s \cos\theta_1$. At this moment the observer receives a note frequency ν' given by the equation (1) as

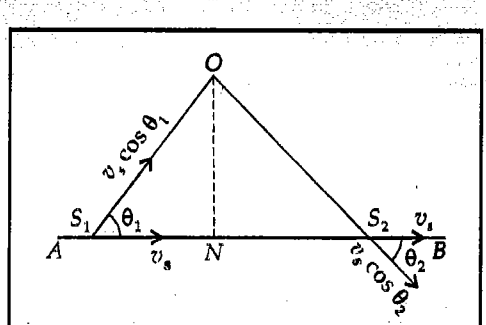


***Figure* 5.6:** Stationary observer not in line with the moving source

$$\nu' = \frac{c}{c - \upsilon_s \cos\theta_1}\nu \qquad \ldots (8)$$

where ν is the original frequency and c is the velocity of sound in the medium which is assumed to be stationary.

As the source approaches N, the foot of the perpendicular from O on AB, θ_1 increases continuously. Consequently, $\cos\theta_1$ decreases and the frequency decreases continuously. When the source is at N, $\theta_1 = \pi/2$ and $\nu' = \nu$.

When the source is at S_2 : the velocity component $\upsilon_s \cos\theta_2$ is away from the observer and hence from Equation(2) the apparent frequency becomes smaller and is given by

$$\nu'' = \frac{c}{c + \upsilon_s \cos\theta_2}\nu \qquad \ldots (9)$$

When the source is act and, θ_2 decreases continuously and ultimately at large distance θ_2 tends to zero.

5.2.6 Observer moving not in line with the stationary source:

Suppose the stationary source S is situated away from the line of motion AB of the observer (Figure-5.7). When the observer is at O_1 the component of his velocity (υ_0) along O_1S is $\upsilon_0 \cos\theta_1$. At this point we may

consider that an observer is moving towards the source with a velocity $\upsilon_0 \cos\theta_1$. Hence the apparent frequency as perceived by the observer would be

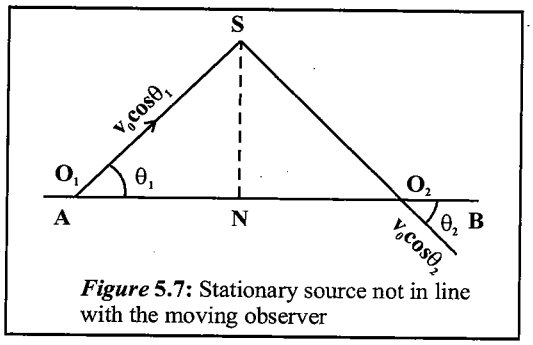


***Figure* 5.7:** Stationary source not in line with the moving observer

$$\nu' = \frac{c + \upsilon_0 \cos\theta_1}{\lambda} = \frac{c + \upsilon_0 \cos\theta_1}{c}\nu \qquad \ldots. (10)$$

where ν is the original frequency of the source and c is the velocity of sound in the medium which is assumed to be stationary.

As the observer approaches N, the foot of the perpendicular from S on AB, θ_1 increases continuously. Consequently $\cos\theta_1$ and hence ν' decreases continuously. At , $\theta_1 = \pi/2$ and $\nu = \nu'$ i.e., the observer now hears a note of the same frequency as emitted by the source. On the other side of N when the observer is at O_2 the velocity component $\upsilon_0 \cos\theta_2$ is along is along SO_2 i.e., away from the source. So the frequency now appears to be

$$\nu' = \frac{c - \upsilon_0 \cos\theta_2}{c}\nu \qquad \ldots (11)$$

In this case $\nu' < \nu$. When the observer moves to a large distance from N then $\theta_2 \to 0$ and the formula (11) reduces to (4).

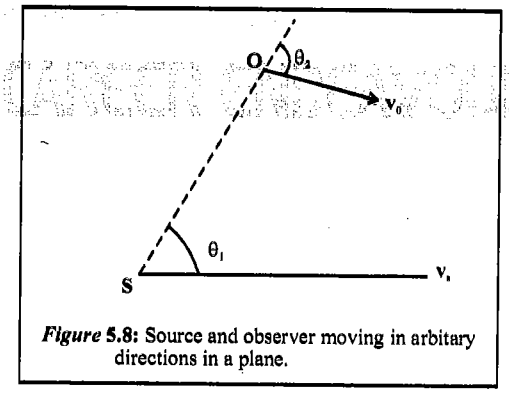


***Figure* 5.8:** Source and observer moving in arbitary directions in a plane.

5.2.7 Source and observer moving in any direction in a plane:

Suppose the source is moving with velocity υ_s in a direction making an angle θ_1 with the line SO joining the source S and observer O at any instant. Let at the same instant the direction of the velocity v_0 of the observer makes an angle θ_2 with the same line SO as shown in Figure-5.8. We may consider the situation to be equivalent to the case where the source is moving towards the observer with a velocity $\upsilon_s \cos\theta_1$ and the observer is moving away from the source with a velocity $\upsilon_0 \cos\theta_2$ and hence the apparent frequency is given by

$$\nu' = \frac{c - \upsilon_0 \cos\theta_2}{c - \upsilon_s \cos\theta_1}\nu \qquad ...(12)$$

5.3 INTERCONNECTION BETWEEN DOPPLER EFFECT AND STATIONARY WAVES

Suppose we have two identical stationary sources S_1 and S_2 emitting sound of same frequency ν and a listener O is moving along the line S_1S_2 as shown in Figure 5.9. As the listener is moving away from the source S_1 with a velocity υ_0 the frequency of the sound from S_1 as perceived by the listener would be

$$\nu_1 = \nu.\frac{c - \upsilon_0}{c} \qquad ... (13)$$

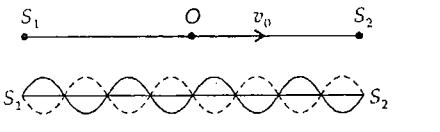


***Figure* 5.9:** Doppler effect and stationary waves

Again as the observer is moving towards S_2, the apparent frequency of the sound from S_2 will be

$$\nu_2 = \nu.\frac{c + \upsilon_0}{c} \qquad ... (14)$$

These notes of frequencies ν_1 and ν_2 will produce beats of frequency

$$\nu_2 - \nu_1 = \nu.\frac{2\upsilon_0}{c} = \frac{2\upsilon_0}{\lambda} \qquad ... (15)$$

where $\lambda = c/\nu$ is the wavelength of the emitted note.

Alternatively, we may think that the listener is moving through a system of stationary waves. The distance between successive nodes or antinodes is $\lambda/2$. Therefore, the number of minimum or maximum sound the listener hears $\upsilon_0/(\lambda/2) = 2\upsilon_0/\lambda$ which agrees with the result obtained from Doppler's principle. Thus we find an interesting connection between Doppler effect, stationary waves and the phenomenon of beats.

SOLVED EXAMPLES

1. A plane wave of frequency ω propagates so that planes of constant phase move with speeds v_1, v_2 and v_3 along the x, y and z axes, respectively. The wave vector, K of the plane wave (in terms of the Cartesian unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$) is:

(a) $K = \omega\left(\frac{1}{v_1}\hat{i} + \frac{1}{v_2}\hat{j} + \frac{1}{v_3}\hat{k}\right)$ (b) $K = \omega\left(\hat{i} + \hat{j} + \hat{k}\right)/\sqrt{v_1^2 + v_2^2 + v_3^2}$

(c) $K = \omega\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)/\left(v_1^2 + v_2^2 + v_3^2\right)$ (d) $K = \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)/\omega$

[JNU : 2010]

Soln. **(c)**

$$\overline{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

$$\vec{k} = \frac{\omega}{|\vec{v}|}(\hat{v}) = \frac{\omega}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \frac{\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)}{\sqrt{v_1^2 + v_2^2 + v_3^2}}$$

$$\boxed{\vec{k} = \frac{\omega\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)}{v_1^2 + v_2^2 + v_3^2}}.$$

Consider two cases (i) a source is approaching a stationary observer with a relative velocity υ and (ii) an observer is approaching a stationary source with the same relative velocity υ. Will the Doppler shift be the same in the two cases?

ln. For the case (i) the apparent frequency is

$$\nu_1 = \nu\frac{c}{c - \upsilon}$$

where ν is the actual frequency and c is the speed of sound.
For the case (ii) the apparent frequency is

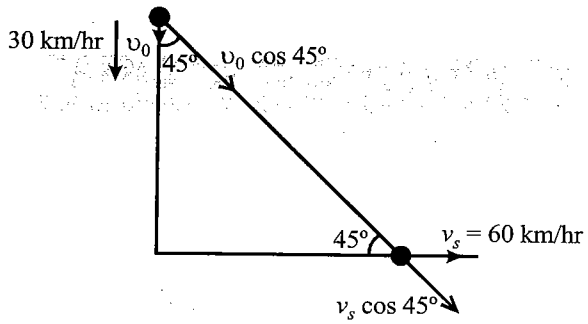
$$\nu_2 = \nu.\frac{c + \upsilon}{c}, \quad \therefore \frac{\nu_2}{\nu_1} = 1 - \frac{\upsilon^2}{c^2}$$

Thus unless υ approaches c, the two frequency will be different.

Two roads meet at right angles to each other. A motor cyclist is moving away from the junction along one road with a speed 60 km/hr and blowing a horn of frequency 200 Hz. Along the other road a bi-cyclist is approaching the first road at speed of 30 km/hr. When they are at equal distances from the junction, what would be the frequency as heard by the bi-cyclist? Given the velocity of sound = 340 m/s.

ln. The required apparent frequency

$$\nu' = \nu.\frac{c - \upsilon_0 \cos\theta_2}{c - \upsilon_s \cos\theta_1}$$



Here $\nu = 200\,\text{Hz}, \theta_1 = \theta_2 = 45°, \upsilon_0 = 30\,\text{km/hr}, \upsilon_s = 60\,\text{km/hr}$

and $\quad c = 340\,\text{m/s} = \frac{340}{1000} \times 60 \times 60\ \text{km/hr} = 1224\,\text{km/hr}$

$$\therefore \quad \nu' = 200\frac{1224 - 30/\sqrt{2}}{1224 - 660/\sqrt{2}} = 203.6\ \text{Hz}$$

A traffic police is blowing a whistle of frequency 200 Hz. If the sound is reflected from an oncoming double decker bus with relative (speed 30 km/hr) what would be the number of beats heard by a stationary observer? Assume the velocity of sound = 340 m/s.

Soln.. The apparent frequency of the reflected sound is given by $\nu' = \nu\dfrac{c+\upsilon_r}{c-\upsilon_r}$

where v_r is relative speed between source and reflecting object

Here $\nu = 200\,\text{Hz}$, $\upsilon_r = 30\,\text{km/hr}$ and $c = \dfrac{340}{1000}\times 60\times 60 = 1224\ \text{km/hr}$

$$\therefore \qquad \nu' = 200\frac{1224+30}{1224-30} = 210\ \text{Hz}$$

Therefore, number of beats heard per second = 210 – 200 = 10.

5. A train is blowing a whistle while passing by a stationary observer. The frequency of sound appears to decrease from 220 Hz to 190 Hz as the train crosses the observer. Find the actual frequency of the whistle.

Soln. If ν is the actual frequency and v_s is the speed of the train then observed frequency would be

$$\nu' = \nu\frac{c}{c-\upsilon_s} \text{ and } \nu'' = \nu\frac{c}{c+\upsilon_s} \qquad \text{where c is the speed of sound}$$

$$\therefore \qquad \frac{\nu'}{\nu''} = \frac{c+\upsilon_s}{c-\upsilon_s} = \text{or, } \frac{\nu'+\nu''}{\nu'-\nu''} = \frac{2c}{2\upsilon_s} = \frac{c}{\upsilon_s}$$

Using this we can write

$$\nu = \nu'\frac{c-\upsilon_s}{c} = \nu'\left(1-\frac{\upsilon_s}{c}\right) = \nu'\left(1-\frac{\nu'-\nu''}{\nu'+\nu''}\right) = \frac{2\nu'\nu''}{\nu'+\nu''} = \frac{2\times 220\times 190}{220+190}\ \text{Hz} = 203.9\ \text{Hz}$$

6. A prominent spectral line of a certain substance has a wavelength 5000 A in the laboratory. The same spectral line is identified in the light coming from a distant star but with a different wavelength 5001 A. Is the star receding from or moving towards to earth? What is speed with respect to the earth?

Soln. As the wavelength increases, the frequency decreases which indicates that the star is receding from the earth Now, frequency of the light coming from the star moving away from the earth with a speed υ will be given by

$$\nu' = \nu.\frac{c}{c+\nu} \quad \text{where } \nu \text{ is the actual frequency.} \qquad \Rightarrow \frac{c}{\lambda'} = \frac{c}{\lambda}.\frac{c}{c+\upsilon}$$

$$\text{or,} \qquad 1+\frac{\upsilon}{c} = \frac{\lambda'}{\lambda} = \frac{5001}{5000} = 1+\frac{1}{5000} \qquad \Rightarrow \upsilon = \frac{1}{5000}c = \frac{1}{5000}\times 3\times 10^8\,\text{m/s} = 6\times 10^4\,\text{km/s}.$$

Chapter 6

Fermat s Principle

6.1 OPTICAL PATH:

When light travels a distance l in a medium of refractive index n the product nl is termed the **optical path** corresponding to the distance l. By definition,

$$n = \frac{\text{velocity of light in vacuum}(c)}{\text{velocity of light in medium}(\upsilon)} \qquad \ldots (1)$$

Therefore optical path,

$$nl = \frac{c}{\upsilon}.l = c.\frac{l}{\upsilon} = c \times \text{time taken by the light to travel the distance } l \text{ in the medium.}$$

$$\ldots (2)$$

6.2 FERMAT'S PRINCIPLE:

The path followed by a ray of light in moving from one point to another point after any number of reflections or refractions would always be stationary (i.e., either maximum or minimum) with respect to variations of that path.

If the light from one point reaches another point directly or after reflection or refraction at a plane surface the path is minimum. Hence time taken by light in this case must also be minimum. Thus the principle of least path may also be called the principal of least time. If the reflection or refraction occurs at spherical surfaces the path may be either maximum or minimum, depending on the curvature of the surfaces. Hence, in general, the path would be stationary.

The optical path between two points may be expressed as $\sum nl$ or $\int ndl$ (if the refractive index n varies continuously over the path). From Fermat's principle,

$$\delta\sum nl = \delta\int ndl = 0 \qquad \ldots (3)$$

6.2.1 Rectilinear propagation of light:

If light moves from one point to another in a homogeneous medium, Fermat's principle implies that path of the ray between the two points should be least. Obviously the path is a straight line. Thus the rectilinear propagation of light results as direct consequence of Fermat's principle.

6.2.2 Principle of reversibility of light:

Fermat's principle stated in the form of the Equation (3), involves only the optical path and not the direction of propagation of light. So a ray in going from the point P to Q will trace the same route as one from Q to P. Thus the principle of reversibility is also included in Fermat's principle.

6.3 TO ESTABLISH LAWS OF REFLECTION AT A SPHERICAL SURFACE FROM FERMAT'S PRINCIPLE:

Suppose a ray of light from a fixed point P is arriving at another fixed point Q via the point A of a spherical surface as shown in Figure 6.1. Let ϕ_1 and ϕ_2 be the angles which the rays PA and AQ make with the normal CA at A drawn from the centre of curvature C of the spherical surface. Let OC = R = radius of the spherical surface. Now from ΔAPC and ΔAQC we can write

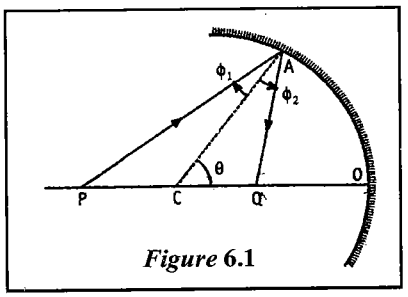


Figure 6.1

$$PA = \left(R^2 + CP^2 + 2R.CP.\cos\theta\right)^{\frac{1}{2}} \quad \text{and} \quad AQ = \left(R^2 + CQ^2 - 2R.CQ.\cos\theta\right)^{\frac{1}{2}} \qquad \ldots (4)$$

According to Fermat's principle the total optical path PA + AQ between P and Q should be stationary and hence the variation of PA + AQ with the angle θ must be zero. Thus,

$$\frac{d}{d\theta}(PA + AQ) = 0$$

Since R, CP and CQ are constants we get from above Equation, using Equations(4)

$$-\frac{CP}{PA} + \frac{CQ}{AQ} = 0 \quad \Rightarrow \quad -\frac{\sin\phi_1}{\sin(180^\circ - \theta)} + \frac{\sin\phi_2}{\sin\theta} = 0$$

or, $\phi_1 = \phi_2$ i.e., angle of incidence (ϕ_1) = angle of reflection (ϕ_2) ... (5)

6.3.1 Reflection formula:

Under paraxial approximation the angle θ as indicated in Figure 6.1 is small so that we may use

$$\cos\theta = 1 - \frac{\theta^2}{2} \qquad \ldots (6)$$

Now assuming OP = x and OQ = y as the magnitude of distances we can write from Equation (4),

$$PA = \left[R^2 + (x - R)^2 + 2R(x - R)\left(1 - \frac{\theta^2}{2}\right)\right]^{\frac{1}{2}}$$

$$= \left[x^2 - R(x - R)\theta^2\right]^{\frac{1}{2}} \simeq x\left[1 - \frac{R(R - x)^2\theta^2}{2x^2}\right]$$

$$= x - \frac{1}{2}R^2\left(\frac{1}{R} - \frac{1}{x}\right)\theta^2 \qquad \ldots (7)$$

Similarly we get from Equation (4),

$$AQ \simeq y - \frac{1}{2}R^2\left(\frac{1}{R} - \frac{1}{y}\right)\theta^2 \qquad \ldots (8)$$

Now for the optical path $L_{op} = PA + AQ$ to an extremum we must have $\frac{dL_{op}}{d\theta} = \frac{d}{d\theta}(PA + AQ) = 0$

Using Equation (7) and (8) we get, $\left[\frac{2}{R} - \frac{1}{x} - \frac{1}{y}\right]\theta = 0 \qquad \ldots (9)$

If the quantity within the bracket is not zero then we must have $\theta = 0$. In this case light falls normally on the surface along PO and gets reflected normally back to Q along OQ. Laws of reflection is obeyed here. On the other hand, if $\frac{1}{y} + \frac{1}{x} = \frac{2}{R} \qquad \ldots (10)$

$\frac{dL_{op}}{d\theta}$ would vanish for all values of θ. This indicates that all paths like PAQ are allowed ray paths i.e., all paraxial rays emanating from P will, after reflection, pass through Q. Therefore, in this case Q will be the image point of the object point P. Thus Equation (10)is reflection formula at the concave spherical surface. Using the sign convention that all the distances measured to the left of O are negative we may write object distance $OP = u = -x$, image distance $OQ = \upsilon = -y$ and radius of curvature $OC = r = -R$. Then formula (10) may be rewritten for a concave spherical reflector as

$$\frac{1}{\upsilon} + \frac{1}{u} = \frac{2}{r} \qquad \ldots (11)$$

6.4 TO ESTABLISH LAWS OF REFRACTION AT A SPHERICAL SURFACE FROM FERMAT'S PRINCIPLE:

Suppose a ray of light from a fixed point Q via the point A on a spherical surface separating two media of refractive indices n_1 and n_2. Let ϕ_1 and ϕ_2 be the angles which the rays PA and AQ make with the normal CA at A drawn from the centre of curvature C of the spherical surface. From ΔAPC and ΔAQC we can write

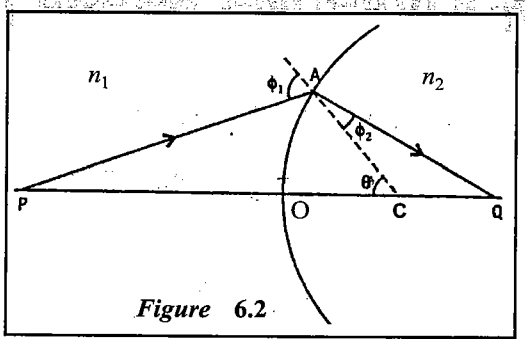


Figure 6.2

$$PA = \left(AC^2 + CP^2 - 2.AC.CP.\cos\theta\right)^{\frac{1}{2}} \text{ and } AQ = \left(AC^2 + CQ^2 + 2.AC.CQ.\cos\theta\right)^{\frac{1}{2}} \qquad \ldots (12)$$

According to Fermat's principle the total optical path $n_1PA + n_2AQ$ between P and Q should be stationary and hence

$$\frac{d}{d\theta}(n_1PA + n_2AQ) = 0 \qquad \ldots (13)$$

Since AC, CP and CQ are constants we get from Equation (13). Using Equations (12).

$$\frac{n_1 CP}{PA} - \frac{n_2 CQ}{AQ} = 0 \quad \Rightarrow \quad \frac{n_1 \sin(180^\circ - \phi_1)}{\sin\theta} - \frac{n_2 \sin\phi_2}{\sin(180^\circ - \theta)} = 0$$

$$\Rightarrow \quad n_1 \sin\phi_1 = n_2 \sin\phi_2 \text{ which is the snell's law of refraction.} \qquad \ldots (14)$$

6.4.1 Refraction formula:

Under paraxial approximation the angle θ as indicated in (Figure 6.2) is small so that we can use

$$\cos\theta = 1 - \frac{\theta^2}{2} \qquad \ldots (15)$$

Now putting OP = x, OQ = y and AC = OC = R as the magnitude of the distances we can write from Equation (12),

$$PA = \left[R^2 + (x+R)^2 - 2R.(x+R)\left(1 - \frac{\theta^2}{2}\right)\right]^{\frac{1}{2}}$$

$$= \left[x^2 + R(x+R)\theta^2\right]^{\frac{1}{2}} \simeq x\left[1 + \frac{R(x+R)\theta^2}{2x^2}\right] = x + \frac{1}{2}R^2\left(\frac{1}{x} + \frac{1}{R}\right)\theta^2$$

$$\ldots (16)$$

Similarly from Equation (12) we get,

$$AQ = y + \frac{1}{2}R^2\left(\frac{1}{y} - \frac{1}{R}\right)\theta^2 \qquad \ldots (17)$$

Now for the optical path $L_{op} = n_1 PA + n_2 AQ$ be an extremum we must have

$$\frac{dL_{op}}{d\theta} = \frac{d}{d\theta}\left[n_1 PA + n_2 AQ\right] = 0$$

$$\Rightarrow \quad R^2\left[n_1\left(\frac{1}{x} + \frac{1}{R}\right) + n_2\left(\frac{1}{y} - \frac{1}{R}\right)\right]\theta = 0$$

$$\Rightarrow \quad \left[\frac{n_2}{y} + \frac{n_1}{x} - \frac{n_2 - n_1}{R}\right]\theta = 0 \qquad \ldots (18)$$

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If the quantity within the bracket is not zero then must have $\theta = 0$. In this case light from P falls on the surface normally along PO and proceeds undeviated to the point Q, obeying the laws of refraction. On the other hand, if

$$\frac{n_2}{y} + \frac{n_1}{x} - \frac{n_2 - n_1}{R} = 0 \qquad \ldots (19)$$

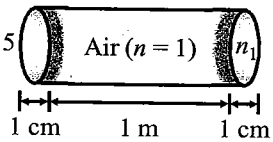
$\frac{dL_{op}}{d\theta} = 0$ for all values of θ. This indicates that all paths like PAQ are allowed ray paths i.e., all paraxial rays emanating from P will, after refraction, pass through Q. Therefore, Q in that case will be the image point of the object point P. Equation (19) then represents the refraction formula at the curved surface.

Using the sign convention that all distances measured to the left of O are negative and those to its right positive we may write object distance OP = u = – x, image distance OQ = v = y and radius of curvature OC = r = R. Then formula 19 may be rewritten for a convex spherical surface in form

$$\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{r} \qquad \ldots (20)$$

SOLVED EXAMPLES

A straight hollow pipe 1m long is closed at both ends with glass plates of 1 cm thick. If the pipe is evacuated what would be the optical path-length between the two outer glass surfaces. Take the refractive index of glass = 1.5.

oln. 

Total optical path

S = (actual path) · (refractive index)

$= (1.5 \times 0.01 + 1 \times 1 + 1.5 \times 0.01)\,\text{m} = 1.03\,\text{m}.$

If the pipe in the above problem is filled with a liquid of refractive index 1.33, find the change in optical path.

oln. In this case total optical path $S_2 = (1.5 \times 0.01 + 1.33 \times 1 + 1.5 \times 0.01)\,\text{m} = 1.36\,\text{m}$

Therefore, increase in optical path is $S_2 - S_1 = 0.33\,\text{m}.$

Consider a refracting surface of radius of curvature r with 'O' as the pole. P and Q are two points such that PO = OQ = x and the centre of curvature lies on the line PQ. Show that the straight optical path POQ is an extremum and this extremum is a minimum when (i) $x < r.\frac{n+1}{n-1}$, (ii) a maximum when $x > r.\frac{n+1}{n-1}$ and (iii) stationary when $x = r.\frac{n+1}{n-1}$.

Given $n = n_2 / n_1 > 1$, where n_1 and n_2 are refractive indices of media on the left and on the right of the refracting surface.

oln.. From Equations (16) and (17) total optical path

$$L_{op} = n_1 PA + n_2 AQ = n_1 x + \frac{1}{2} n_1 r^2 \left(\frac{1}{x} + \frac{1}{r}\right)\theta^2 + n_2 y + \frac{1}{2} n_2 r^2 \left(\frac{1}{y} - \frac{1}{r}\right)\theta^2$$

where we put PO = OQ = x, R = r

Now $\frac{dL_{op}}{d\theta} = r^2\left[n_1\left(\frac{1}{x} + \frac{1}{r}\right) + n_2\left(\frac{1}{x} - \frac{1}{r}\right)\right]\theta$

For $\theta = 0$, L_{op} becomes the strainght optical path POQ and $\frac{dL_{op}}{d\theta} = 0$

Thus the straight path POQ is an extremum.

Now $\frac{d^2L_{op}}{d\theta^2} = r^2\left[n_1\left(\frac{1}{x} + \frac{1}{r}\right) + n_2\left(\frac{1}{x} - \frac{1}{r}\right)\right]$ $\Rightarrow \frac{d^2L_{op}}{d\theta^2} = r^2.n_1\left[(n+1).\frac{1}{x} - (n-1)\frac{1}{r}\right]$

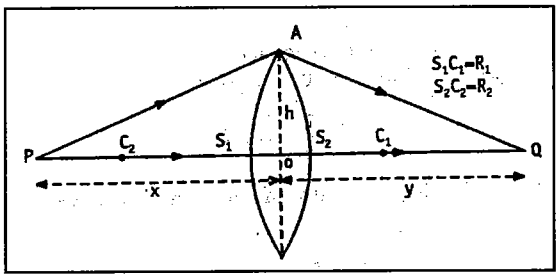
where we put $n = n_2 / n_1$.

For L_{op} to be maximum, $\frac{d^2L_{op}}{d\theta^2} < 0$ $\Rightarrow (n-1).\frac{1}{r} > (n+1).\frac{1}{x}$ $\Rightarrow x > \frac{n+1}{n-1}.r$

For L_{op} to be minimum, $\frac{d^2L_{op}}{d\theta^2} > 0 \quad \Rightarrow \quad x < r.\frac{n+1}{n-1}$

For L_{op} to be stationary, $\frac{d^2L_{op}}{d\theta^2} = 0 \quad \Rightarrow \quad x = r.\frac{n+1}{n-1}$

4. Establish thin lens formula by using Fermat's principle.



Soln.. Let in figure above, Q be the image point of an axial object at point P. According to Fermat's principle the two optical paths from P to Q as shown in figure must be same i.e.,

$$PA + AQ = PS_1 + n.S_1S_2 + S_2Q \qquad \ldots (1)$$

Now from above figure

$$PA = \left(x^2 + h^2\right)^{\frac{1}{2}} \simeq x + \frac{1}{2}\frac{h^2}{x}$$

$$AQ = \left(y^2 + h^2\right)^{\frac{1}{2}} \simeq y + \frac{1}{2}\frac{h^2}{y}$$

where we assume *h* small (i.e., paraxial approximation).

Therefore, from Equation (1)

$$x + y + \frac{1}{2}h^2\left(\frac{1}{x} + \frac{1}{y}\right) = PO - OS_1 + n\left(OS_1 + OS_2\right) + OQ - OS_2$$

$$= x + y + (n-1)(OS_1 + OS_2)$$

CAREER ENDEAVOUR

Therefore, $\frac{1}{x} + \frac{1}{y} = (n-1).\frac{2}{h^2}.(OS_1 + OS_2) \qquad \ldots (2)$

Now for the spherical surface S_1 with the centre of curvature at C_1, we can write from geometry

$$h^2 = OS_1\left(2S_1C_1 - OS_1\right)$$

For a thin lens $OS_1 << S_1C_1$ and we can write

$$h^2 = OS_1.2R_1$$

Similarly for the surface S_2,

$$h^2 = OS_2.2R_2$$

substituting these values of h^2 in Equation (2) we get

$$\frac{1}{x} + \frac{1}{y} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \qquad \ldots (3)$$

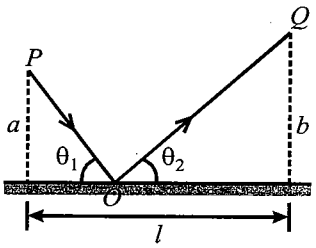
Now according to our sign convention, object distance OP = u = - x, image distance OQ = υ = y, radius of curvature of the first surface S_1, $S_1C_1 = r_1 = +R_1$, radius of curvature of the second surface S_2, $S_2C_2 = r_2 = -R_2$. Therefore, formula (2) takes the usual form

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f} \qquad \ldots (4)$$

where f, known as the focal length of the lens, is given by

$$\frac{1}{f}=(n-1)\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \qquad \ldots (5)$$

Consider a ray of light starting from point P in front of mirror and going to another point Q after getting reflected from the mirror, as shown in the figure.



Show that the incident and reflected light make same angle from the mirror using Fermat principal.

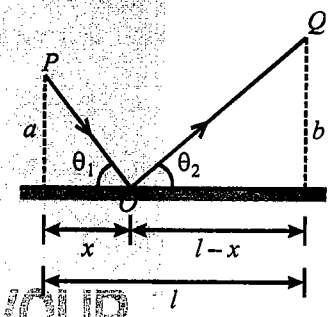
ln. Let O is at x distance from point M. Now, total path travel by the light ray

$$S = PO + OQ = \sqrt{a^2+x^2}+\sqrt{b^2+(l-x)^2}$$

Now, according to Fermat principal, path followed by the light will be minimum, so that time taken is the least. Thus, time taken will be least, if path S is minimum.

$$\frac{dS}{dx}=\frac{d(a^2+x^2)^{1/2}}{dx}+\frac{d[b^2+(l-x)^2]^{1/2}}{dx}=0$$

$$=\frac{x}{(a^2+x^2)^{1/2}}-\frac{(l-x)}{\sqrt{b^2+(l-x)^2}}=0$$



from given figure, we can write $\cos\theta_1 - \cos\theta_2 = 0$

$\Rightarrow \cos\theta_1 = \cos\theta_2$

$\Rightarrow \theta_1 = \theta_2$

CAREER ENDEAVOUR

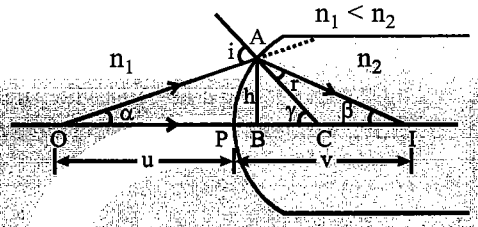
i.e., incident angle is equal to the reflected angle.

Chapter 7

Refraction at Spherical Surfaces

7.1 REFRACTION AT SPHERICAL SURFACES:

Consider a convex spherical surface separating two media of refractive index n_1 and n_2 $(n_2 > n_1)$.



Ray diagram for the real image is shown in above figure where O is the point object and I is the image formed by the spherical surface.

Object distance u, image distance v and radius of curvature r is related by the formula

$$\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{r} \quad \ldots (1)$$

This equation is known as **Gaussian formula** for the single refracting surface.

Here, $\frac{n_2 - n_1}{r}$ is called power of the refracting surface.

If r is expressed in meter, unit of power is dioptre (D). Above formula is applicable for refraction at concave surface also.

CAREER ENDEAVOUR

7.1.1 Principal foci of a spherical surface:

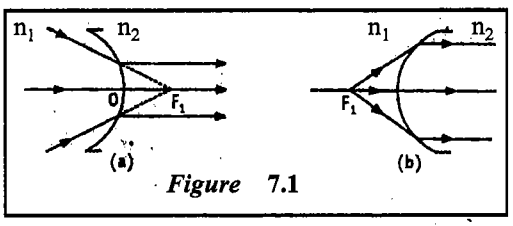


Figure 7.1

First principal focus : The first focal point (F_1) is a point on the principal axis such that rays moving towards that point [Figure 7.1(a)] or diverging from that point [Figure 7.1(b)] become parallel to the principal axis after refraction. *The distance of F_1 from the pole of the refracting surface is called the first principal focal length* (f′) *of the surface.* When $f' = u$, then $v = \infty$. From the general formula (1)

$$f' = -\frac{n_1 r}{n_2 - n_1} \quad \ldots (2)$$

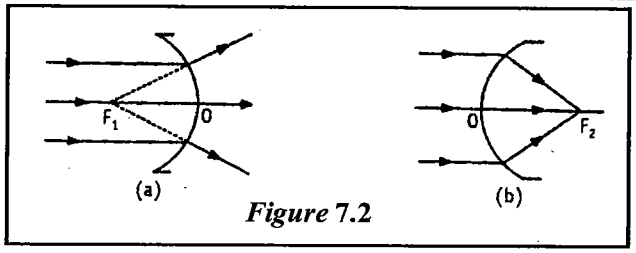


Figure 7.2

Second principal focus : The second focal point (F_2) is a point on the principal axis such that the incident rays moving parallel to the principal axis, after refraction, either appear to diverge from this point [Figure 7.2(a)] or actually converge to this point [Figure 7.2(b)].

This distance of F_2 from the pole of the refracting surface is called the second principal focal length (f) of the surface. When $f = v$, then $u = \infty$. From the general formula (1),

$$f = \frac{n_2 r}{n_2 - n_1} \qquad \ldots (3)$$

From Equation (2) and (3) we get, $\frac{f}{f'} = -\frac{n_2}{n_1}$ ... (4)

This indicates that the two principal focal lengths are always of opposite signs.

7.2 MAGNIFICATION FOR A SPHERICAL SURFACE:

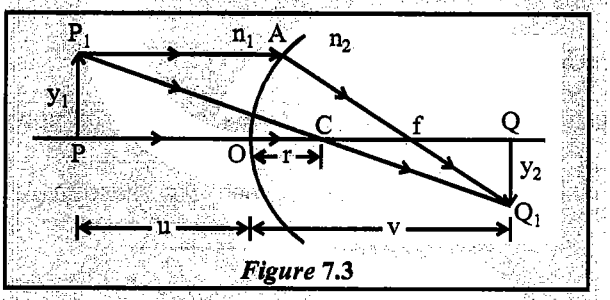


Figure 7.3

In Figure 7.3 let PP_1 be an object and QQ_1 be its image formed by refraction at the concave spherical surface separating two media of refractive indices n_1 and n_2.

CAREER ENDEAVOUR

7.2.1 Lateral or linear magnification:

Linear magnification (m) is defined as the ratio of the image height (y_2) to the object height (y_1) both measured perpendicular to the axis. Thus,

$$m = \frac{y_2}{y_1} = \frac{QQ_1}{PP_1} = \frac{CQ}{CP} = \frac{OQ - OC}{OP - OC} \qquad \ldots (5)$$

$[\because \Delta PP_1C$ and ΔQQ_1C are similar$]$

Now according to our sign convention all distances measured to the left of O are negative. Let the object distance u = – OP, image distance v = –OQ and the radius of curvature r = –OC.

$$m = \frac{v - r}{u - r} \qquad \ldots (6)$$

From the relation, $\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{r}$ we get,

$$n_1\left(\frac{1}{r}-\frac{1}{u}\right)=n_2\left(\frac{1}{r}-\frac{1}{v}\right) \Rightarrow n_1\frac{u-r}{u}=n_2\frac{v-r}{v} \quad \ldots (7)$$

Hence from Equation (6) and (7), $m=\frac{n_1}{n_2}.\frac{v}{u}$ ... (8)

7.2.2 Longitudinal magnification:

If a small object of length δu, placed along the axis produces an images of length δv, the longitudinal magnification (l) is defined as

$$l=\lim\frac{\delta v}{\delta u}=\frac{dv}{du}.$$

Differentiating the equation, $\frac{n_2}{v}-\frac{n_1}{u}=\frac{n_2-n_1}{r}$ w.r.t. u, we get

$$-\frac{n_2}{v^2}.\frac{dv}{du}+\frac{n_1}{u^2}=0 \Rightarrow \frac{dv}{du}=\frac{n_1}{n_2}.\frac{v^2}{u^2}$$

From equation (8)

$$\Rightarrow \quad l=\frac{dv}{du}=\left(\frac{n_1v}{n_2u}\right)^2.\frac{n_2}{n_1}=m^2.\frac{n_2}{n_1} \quad \ldots (9)$$

7.2.3 Angular magnification:

If the incident ray PA and the corresponding refracted ray AB make angles α_1 and α_2 respectively with the axis (Figure 7.3) the ratio $\frac{\tan\alpha_2}{\tan\alpha_1}$ is called the angular magnification (M).

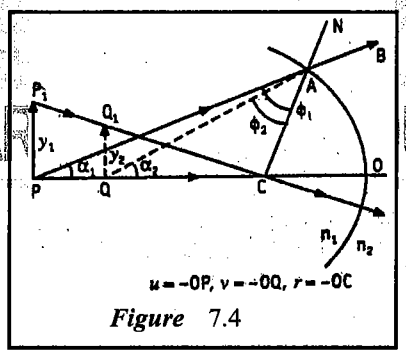


Figure 7.4

$$\Rightarrow \quad M=\frac{\tan\alpha_2}{\tan\alpha_1}=\frac{AO/OQ}{AO/OP}=\frac{OP}{OQ}=\frac{u}{v} \quad \ldots (10)$$

Using Equation (8) we can write, $M=\frac{n_1}{n_2}.\frac{1}{m}$ ... (11)

7.2.4 Inter-relation:

Combining Equation (9) and (11) we get $l=m\times\frac{1}{M} \Rightarrow \frac{l\times M}{m}=1$

$$\Rightarrow \frac{\text{Longitudinal magnification}\times\text{Angular magnification}}{\text{Linear magnification}}=1 \quad \ldots (12)$$

7.3 LAGRANGE'S LAW RELATION:

In the paraxial region there exists an important relation between object size, image size and the angles which the conjugate rays make with the principal axis. Let in Figure 7.4, PP_1 be a small object of size y_1 placed transversely to the axis in front of a certain refracting surface. C is the centre of curvature and QQ_1 is the image of size y_2. The conjugate rays PA and AB make angles α_1 and α_2 respectively with the axis. From Figure 7.4 we can write,

$$\frac{y_2}{y_1}\cdot\frac{\tan\alpha_2}{\tan\alpha_1}=\frac{QQ_1}{PP_1}\times\frac{AO/OQ}{AO/OP}$$ [in the paraxial region the point A is very close to pole O]

Using equation (5)

$$=\frac{CQ}{CP}\times\frac{OP}{OQ}=\frac{OQ-OC}{OP-OC}\times\frac{OP}{OQ} \qquad \ldots (13)$$

According to our sign convention all distances measured to the left of O are negative. Let the object distance $u = -OP$, image distance $\upsilon = -OQ$ and radius of curvature $r = -OC$. Therefore,

$$\frac{y_2}{y_1}\cdot\frac{\tan\alpha_2}{\tan\alpha_1}=\frac{v-r}{u-r}\times\frac{u}{v} \qquad \ldots (14)$$

Now we have the relation

$$\frac{n_2}{v}-\frac{n_1}{u}=\frac{n_2-n_1}{r} \quad\Rightarrow\quad n_1\left(\frac{1}{r}-\frac{1}{u}\right)=n_2\left(\frac{1}{r}-\frac{1}{v}\right)$$

$$\Rightarrow \quad \frac{n_1}{n_2}=\frac{v-r}{u-r}\cdot\frac{u}{v} \qquad \ldots (15)$$

Combining Equation (14) and (15) we get

$$n_1 y_1 \tan\alpha_1 = n_2 y_2 \tan\alpha_2$$

This relation is also called Lagrange's law

Helmholtz's relation : If there is m number of co-axial refractive surface of indices $n_1, n_2, n_3, \ldots, n_m$, then final n_m, y_m and $\tan\alpha_m$ is related by

$n_1 y_1 \tan\alpha_1 = n_m y_m \tan\alpha_m$ ... (16)

CAREER ENDEAVOUR

Its importance lies in the fact that it expresses linear magnification in terms of angular magnification. *Note that this relationship between the magnifications is independent of the refractive indices of the intervening media.*

A system is known as telescopic system if all the cardinal (focal, principal and nodal) points lie at infinity. In this case, usually $n_1 = n_m$. Hence we get

$$\frac{y_1}{y_m}=\frac{\tan\alpha_m}{\tan\alpha_1} \qquad \ldots (17)$$

Thus in a telescopic system angular magnification $(\tan\alpha_m / \tan\alpha_1)$ is equal to the reciprocal of linear magnification (y_m / y_1).

7.4 ABBE'S SINE CONDITION:

$$n_1 y_1 \sin\alpha_1 = n_2 y_2 \sin\alpha_2$$

This is known as Abbe's sine condition. It holds good for all rays whether paraxial or not.

When the rays are paraxial i.e., α_1 and α_2 are small we can write $\sin\alpha_1 \approx \tan\alpha_1$ and $\sin\alpha_2 \approx \tan\alpha_2$. In this case Abbe's sine condition reduces to Helmholtz's relation.

7.5 REFRACTION THROUGH THIN LENS:

7.5.1 Some important terms:

(i) Principal axis: Principal axis of a lens is the straight line joining the centres of curvature of its two bounding surfaces.

(ii) Poles: The points of intersection of the two surfaces with the principal axis are called the poles of the lens. The distance between the two poles is called the thickness of the lens (which might be considered zero for thin lens).

(iii) Principal Section: The section of a lens by a plane through its principal axis is called principal section.

(iv) Aperture: When the bounding line of the lens is circular, the diameter of this circular boundary is called aperture of the lens.

(v) Optical centre: Optical centre of a lens is a point on the principal axis such that any ray through it emerges parallel to its direction of incidence. For equi-convex lens the optical centre is situated at the mid point.

(vi) First focal point: First focal point (F_1) of a lens is a point on the principal axis such that rays moving towards that point [Figure 7.4(a)] or diverging from that point [Figure 7.4(b)] become parallel to the principal axis after refraction. The distance of F_1 from the optical centre (O) of the lens is called the first focal length (f'), when $v = \infty$.

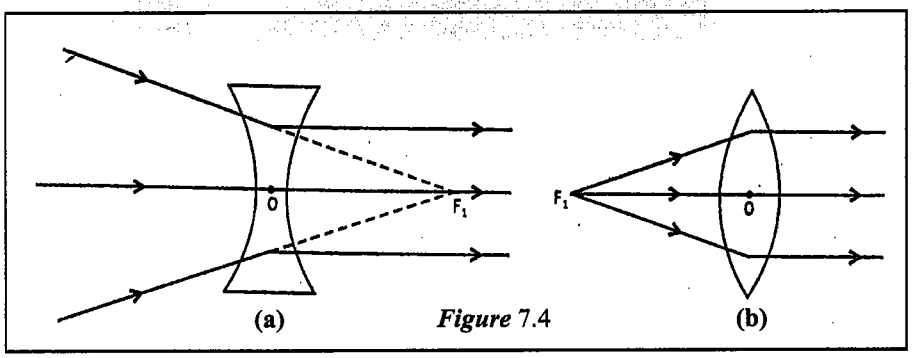


Figure 7.4

(vii) Second focal length: Second focal length (F_2) is a point on the principal axis such that the incident rays moving parallel to the principal axis will, after refraction, either appear to diverge from this point [Figure 7.5(a)] or actually converge to point [Figure 7.5(b)]. The distance of F_2 from the optical centre (O) of the lens is called the second focal length (f). Hence $f = \upsilon$ when $u = \infty$. According to our sign convention f is negative for a diverging lens and positive for a converging lens.

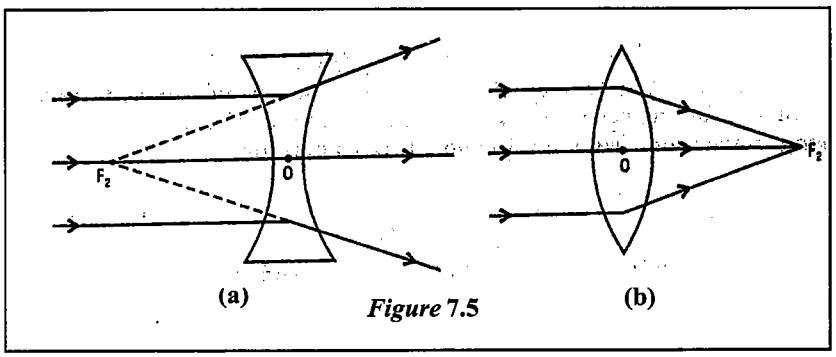


Figure 7.5

(viii) Power: Power of a lens is defined as $P = \frac{1}{\text{focal length}} = \frac{1}{f}$ ... (19)

The unit of power is **dioptre** which is the power of a lens of one metre focal length. The power of a convex lens is considered to be positive and that of a concave lens negative.

Ex. 1. A convex lens of focal length 40 cm is in contact with a concave lens of focal length 25 cm. The power of the combination is

(a) – 1.5 D (b) – 6.5 D (c) + 6.5 D (d) + 6.67 D

[IIT-JEE : 1982]

Soln. (a)

$$P = P_1 + P_2 = \frac{1}{f_1(m)} = \frac{1}{0.4} - \frac{1}{0.25} = -1.5\ D$$

7.5.2 Lens formula with optical centre as origin:

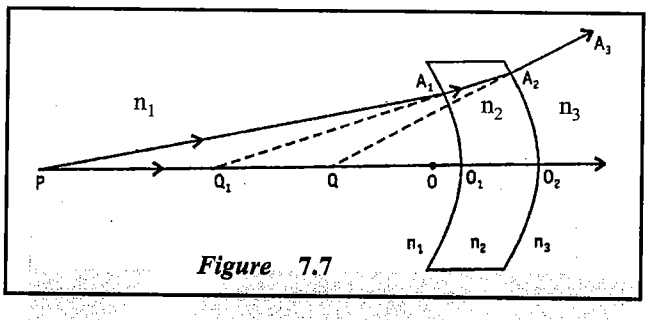


Figure 7.7

$$\frac{n_3}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{r_1} - \frac{n_2 - n_3}{r_2} \quad \ldots (20)$$

$$\frac{n_3}{v} - \frac{n_1}{u} = \frac{1}{F}$$

where, u = object distance, v = image distance ,

$$\frac{1}{F} = \frac{n_2 - n_1}{r_1} - \frac{n_2 - n_3}{r_2} \quad \ldots (21)$$

According to the definition of focal lengths,

u = f '(1st focal length) when v = ∞;

v = f '(2nd focal length) when u = ∞;

CAREER ENDEAVOUR

Putting these conditions in equation (20), we get

$$-\frac{n_1}{f'} = \frac{n_3}{f} = \frac{1}{F} \quad \ldots (22)$$

$$\Rightarrow \quad \frac{f'}{f} = -\frac{n_1}{n_3} = -\frac{\text{R.I. of incident medium}}{\text{R.I. of emergent medium}} \quad \ldots (23)$$

Combining (20) and (22) we get, $\frac{n_3}{v} - \frac{n_1}{u} = \frac{n_3}{f}$ ... (24)

Case-I: When $n_1 = n_3$ $\Rightarrow \frac{1}{f} = -\frac{1}{f'} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$ ... (25)

And $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ ... (26)

Case-II: When the surrounding medium is air, $n_1 = n_3 = 1$ and if $n_2 = n$ then

$$\frac{1}{f} = \frac{1}{f'} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \quad \ldots (27)$$

The power of the lens is $P = \frac{1}{F} = \frac{n_2 - n_1}{r_1} + \frac{n_3 - n_2}{r}$ ... (28)

= Sum of the powers of the refracting surfaces.

7.5.2 Conjugate foci relation with the two foci as origin – Newton's equation:

The conjugate foci relation for a thin lens with optical centre (O) as origin is given by

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} = -\frac{1}{f'} \quad \ldots (29)$$

where we assume that the media on both sides of the lens are same. Therefore, $f' = -f$ and we can write

$$\frac{f}{v} - \frac{f}{u} = 1 \quad \Rightarrow \quad \frac{f}{v} + \frac{f'}{u} = 1 \quad \Rightarrow uv - uf - vf' + ff' = ff'$$

$$\Rightarrow \quad (u - f')(v - f) = ff' \quad \ldots (30)$$

$$UV = ff' \quad \ldots (31)$$

Here U and V represent the object distance and its image distance measured from the first and second foci respectively.

This equation is known as Newton's equation.

Ex. 2. When a point object is placed at a distance $u = 36$ cm from a biconvex lens, the image forms at a distance $v = 18$ cm on the other side of the lens. If the distance of the object is reduced to 12 cm, the image will be formed at

(a) 12 cm (b) ∞ (c) 6 cm (d) $-\infty$

[IIT-JAM GP : 2008]

Soln. **(b)**

$$\frac{1}{f} = \frac{1}{v} - \frac{1}{u} = \frac{1}{18} - \frac{1}{-36} = \frac{1}{12}$$

CAREER ENDEAVOUR

$f = 12\ cm$

Now, $u = 12\ cm$

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\frac{1}{v} - \frac{1}{-12} = \frac{1}{12}$$

$v = \infty$

Ex. 3. A concave lens of glass, refractive index 1.5 has both surfaces of same radius of curvature R. On immersion in a medium of refractive index 1.75, it will behave as a

(a) convergent lens of focal length 3.5 R (b) convergent lens of focal length 3.0 R

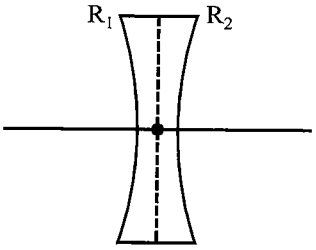
(c) divergent lens of focal length 3.5 R (d) divergent lens of focal length 3.0 R

[IIT-JEE : 1999]

Soln. **(a)**

$R_1 = -R$, $R_2 = +R$, $\mu_g = 1.5$ and $\mu_m = 1.75$

$$\therefore \quad \frac{1}{f}=\left(\frac{\mu_g}{\mu_m}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

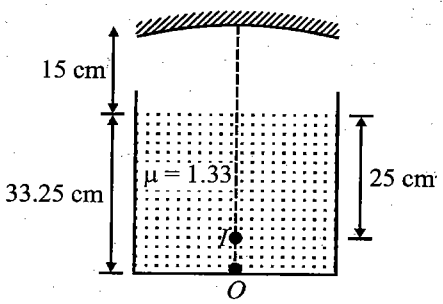


$$\frac{1}{f}=\left(\frac{1.5}{1.75}-1\right)\left(\frac{1}{-R}-\frac{1}{R}\right)=\frac{1}{3.5R}$$

$\therefore \quad f=+3.5R$

Therefore, in the medium it will behave like a convergent lens of focal length 3.5R. It can be understood as, $\mu_m > \mu_g$, the lens will change its behaviour.

Ex. 4. A container is filled with water ($\mu = 1.33$) upto a height of 33.25 cm. A concave mirror is placed 15 cm above the water level and the image of an object placed at the bottom is formed 25 cm below the water level. The focal length of the mirror is



(a) 10 cm (b) 15 cm (c) 20 cm (d) 25 cm

[IIT-JEE : 2005]

Soln. (c)

Apparent depth of the object majored from the top in a medium of refractive index n is given by $h' = \frac{h}{n}$.

Distance of object from mirror $= 15 + \frac{33.25}{1.33} = 40$ cm.

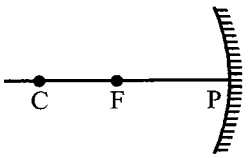
Distance of image from mirror $= 15 + \frac{25}{1.33} = 33.8$ cm.

For the mirror, $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$

$\therefore \quad \frac{1}{-33.8}+\frac{1}{-40}=\frac{1}{f}$

$\therefore \quad f = -18.3$ cm

Ex. 5. In the concave mirror arrangement below C and F stand for the centre of curvature and the focal point respectively. Where on the axis would one place an object for the image to have the same dimension as the object itself?



(a) Midway between F and P (b) At F
(c) Midway between C and F (d) At C

[IISc : 2006]

Ans. (d)

7.6 EQUIVALENT LENS AND EQUIVALENT FOCAL LENGTH:

A single lens is said to be equivalent to a system of lenses when that single lens produces an image of a given object at the same place and of the same size as that formed by the system of lenses. The focal length of such a single lens is called equivalent focal length of the system of lenses.

The equivalent focal length F of two thin lenses having second focal lengths f_1 and f_2 and kept separated by a distance t in air is given by

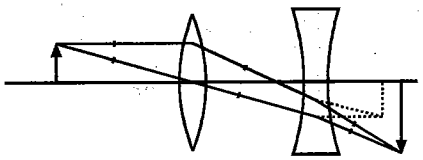
$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{t}{f_1 f_2} \qquad \ldots (32)$$

Ex. 6. Two lenses of power +5D and –2D are in contact and are coaxially situated. The power of the combination is

(a) –3D (b) +3D (c) –7D (d) +7D **[DU 2015]**

Soln. **(b)** $P=P_1+P_2=+5D-2D=+3D$

Ex. 7. A convex lens and a concave lens, each with a focal length of 4 cm are separated by a distance of 6 cm along their axis as shown below :



An object is placed 8 cm before the first lens. Its image is

(a) real and erect (b) virtual and erect (c) real and inverted (d) virtual and inverted

[IISc : 2008]

Soln. **(c)** $u=-8\,cm,\quad f=4\,cm$

$$\frac{1}{v}-\frac{1}{u}=\frac{1}{f}$$

$$\frac{1}{v}+\frac{1}{8}=\frac{1}{4}$$

$$v=8\,cm$$

For concave lens

The image formed by convex lens forms a virtual object

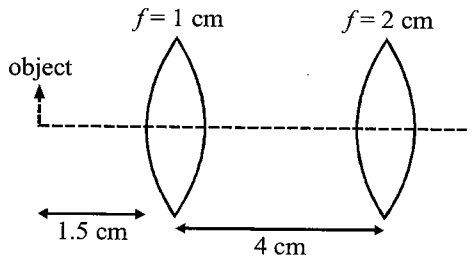
$u=2\,cm,\quad f=-4\,cm$

$$\frac{1}{v}-\frac{1}{2}=\frac{1}{-4}$$

$$v=4\,cm$$

Image is real and inverted.

Ex. 8. Two thin convex lenses L_1 and L_2 with focal lengths 1 cm and 2 cm respectively are separated by a distance of 4 cm along their as shown below:



An object is placed at a distance of 1.5 cm before the first lens. The ratio of the final image size to the object size is

(a) 1 (b) 2 (c) 3 (d) 4

[IISc : 2009]

Soln. **(d)**

For first lens, $u = -1.5, f = 1$ cm

From Len's maker formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \frac{1}{v} - \frac{1}{-1.5} = \frac{1}{1}$$

$$\Rightarrow \frac{1}{v} = 1 - \frac{2}{3} = \frac{1}{3}$$

So, $v = 3\ cm$

Magnification $m_1 = \frac{v}{u} = \frac{3}{-1.5} = -2$

For second lens

$u = -1\ cm, f = 2\ cm$

From Len's maker formula

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$$

$$\Rightarrow \frac{1}{v} - \frac{1}{-1} = \frac{1}{2}$$

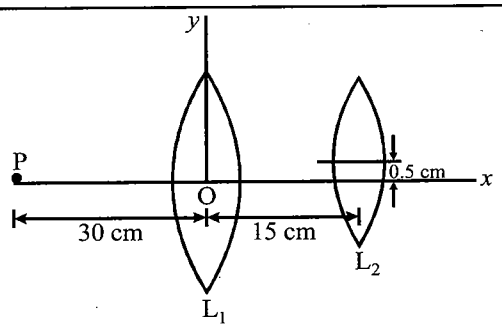
$$\frac{1}{v} = \frac{1}{2} - 1 = -\frac{1}{2}$$

Magnification, $m_2 = \frac{v}{u} = \frac{-2}{-1} = 2$

Therefore, final magnification, $m = m_1 m_2 = -2 \times 2 = -4$

So, Ratio of final image size to the object size = 4.

Ex. 9. Two thin lenses L_1 and L_2 of focal length 15 cm and 10 cm respectively, are kept 15 cm apart from each other. Their axes are separated by 0.5 cm as shown in figure (not to scale). If a point object P is placed on the axis of L_1 to its left at a distance of 30 cm, find the x and y coordinates (origin O) of the image formed by the combination,



[IIT-JAM : 2007]

Soln. For L_1 $\quad u_1 = -30$ cm, $f_1 = 15$ cm; $\quad$ since $\frac{1}{v_1} - \frac{1}{u_1} = \frac{1}{f_1} \Rightarrow v_1 = 30$cm

For L_2, the image of L_1 will act as a virtual object.

$u_2 = 30 - 15 = 15$ m, $\quad f_2 = 10$ cm; $\quad$ since $\frac{1}{v_2} - \frac{1}{u_2} = \frac{1}{f_2} \Rightarrow v_2 = 6$ cm

Magnification, $m = \frac{v_2}{u_2} = \frac{6}{15}$.

The object for L_2 lies on the principal axis of L_1, so, height of object $h = -0.5$ cm,

Magnification, $m = \frac{h_i}{h_0} = \frac{6}{15} \Rightarrow h_i = \frac{6}{15} \times h_0 = \frac{6}{15} \times (-0.5) = -0.2$ cm

Negative sign means image is inverted i.e., below the principal axis of L_2.

x coordinate of image = 15 + 6 = 21 cm

y coordinate of image = 0.5 – 0.2 = 0.3 cm

So, co-ordinate of the image $(21, 0.3)$

Ex. 10. An astronomical telescope has an angular magnification of magnitude 5 for far objects. The separation between the objective and the eyepiece is 36 cm and the final image is formed at infinity. The focal length f_o of the objective and the focal lenght f_e of the eyepiece are

(a) $f_o = 45$ cm and $f_e = -9$ cm (b) $f_o = 50$ cm and $f_e = 10$ cm

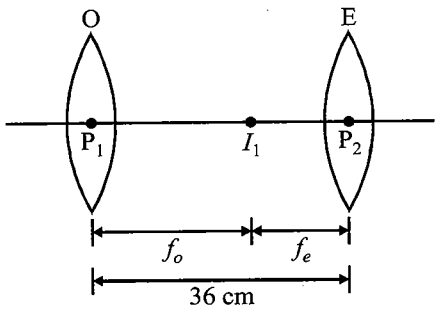
(c) $f_o = 7.2$ cm and $f_e = 5$ cm (d) $f_o = 30$ cm and $f_e = 6$ cm

[IIT-JEE : 1989]

Soln. **(d)**

Image formed by objective (I_1) is at second focus of it because objective is focussed for distant objects.

Therefore, $P_1I_1 = f_o$



Further I_1 should lie at first focus of eyepiece because final image is formed at infinity.

$\therefore \quad P_2I_1 = f_e$

Given $\quad P_1P_2 = 36\ cm$

$\therefore \quad f_o + f_e = 36$ ...(i)

Further angular magnification is given as 5. Therefore, $\frac{f_o}{f_e} = 5$ ...(ii)

Solving equations (i) and (ii), we get $f_o = 30\ cm$ and $f_e = 6\ cm$

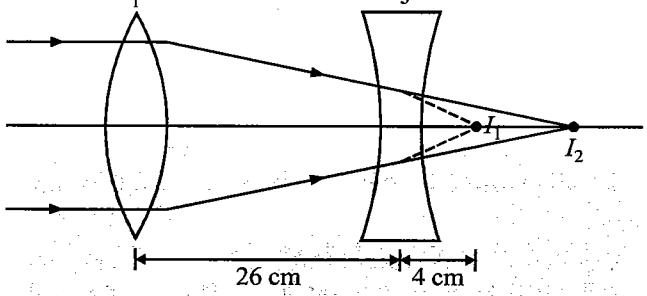
. 11. The size of the image of an object, which is at infinity, as formed by a convex lens of focal length 30 cm is 2 cm. If a concave lens of focal length 20 cm is placed between the convex lens and the image at a distance of 26 cm from the convex lens, calculate the new size of the image.

(a) 1.25 cm (b) 2.5 cm (c) 1.05 cm (d) 2 cm

[IIT-JEE : 2003]

n. (b)

Image formed by convex lens at I_1 will act as a virtual object for concave lens. For concave lens



$\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$ or $\frac{1}{v} - \frac{1}{4} = \frac{1}{-20}$ or $v = 5\ cm$

Magnification for concave lens $m = \frac{v}{u} = \frac{5}{4} = 1.25$. As size of the image at I_1 is 2 cm. Therefore, size of image at I_2 will be $2 \times 1.25 = 2.5$ cm.

7.7 SOME IMPORTANT PROBLEMS WITH LENSES:

7.7.1 Minimum distance between an object and its real image produced by a convex lens:

Suppose we consider a convex lens in air forming real image on its right hand side for an object placed on the left. Here object distance u is –ve, image distance v and focal length f are +ve. So by putting numerical values of u, v and f in the formula $\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \quad \Rightarrow \frac{1}{v} + \frac{1}{u} = \frac{1}{f}$ ... (33)

For the distance $D = u + v$ to be minimum, $\frac{d}{du}(u + v) = 0 \quad \Rightarrow \frac{dv}{du} = -1$

Now from Equation (33), on differentiation, $-\frac{1}{v^2} - \frac{1}{u^2}\frac{du}{dv} = 0$

Therefore, for D to be minimum, $-\frac{1}{v^2} - \frac{1}{u^2}(-1) = 0 \quad \Rightarrow u = v$

Therefore, from Equation (33) $u = v = 2f$

Thus $D_{min} = u + v = 2f + 2f = 4f$

7.7.2 Condition for the formation of two real images of a given object, on the same screen, for two different positions of a convex lens and expression for focal length in terms of lens displacement:

Here the distance between the object and the screen i.e., $D = u + v$ is constant. Therefore, putting $v = D - u$ in Equation (33) we get, $\frac{1}{D-u} + \frac{1}{u} = \frac{1}{f}$

$$\Rightarrow \quad u^2 - Du + Df = 0 \qquad \ldots (34)$$

For real distinct roots of this quadratic equation we must have $D^2 > 4Df \Rightarrow D > 4f$

So, for getting two real images on the screen the distance between the object and the screen must be greater than four times the focal length of the convex lens.

Solving Equation (34) we get for the two positions of the lens

$$u_1 = \frac{D + \sqrt{D^2 - 4Df}}{2} \text{ and } u_2 = \frac{D - \sqrt{D^2 - 4Df}}{2} \qquad \ldots (35)$$

Therefore, displacement of the lens between the two positions is

$$x = u_1 - u_2 = \sqrt{D^2 - 4Df} \quad \Rightarrow \quad f = \frac{D^2 - x^2}{4D} \qquad \ldots (36)$$

7.7.3 Size of the object is the geometric mean of the sizes of the images produced by a convex lens at two positions:

Let u, v and I_1 be respectively the object distance, image distance and length of the image for the first position L_1 of the lens; while the corresponding quantities for the second position L_2 of the lens would be v, u and I_2 ($\because$ the positions of object and image are interchangeable) [See Figure 7.7]. If O be the length of the object then magnifications for the two positions are given by

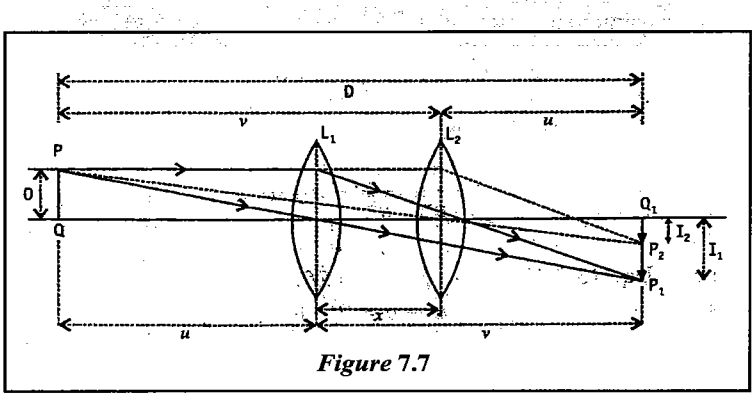


Figure 7.7

$$m_1 = \frac{I_1}{O} = \frac{v}{u} \text{ and } m_2 = \frac{I_2}{O} = \frac{u}{v} \qquad \ldots (37)$$

$$\Rightarrow \quad \frac{I_1 I_2}{O^2} = 1 \quad \Rightarrow \quad O = \sqrt{I_1 I_2} \qquad \ldots (38)$$

7.7.4 Focal length of a convex lens in terms of the linear magnifications of its images:

A convex lens produces a real image on a fixed screen at its two positions when the distance D between the object and screen is greater than $4f$. Let u_1 and v_1 be the numerical values of the object and image distances when the convex lens is nearer to the object while u_2 and υ_2 are the corresponding quantities when the lens is nearer to the screen. From the relation (33) we get,

$$\frac{1}{v_1}+\frac{1}{u_1}=\frac{1}{f} \Rightarrow 1+\frac{v_1}{u_1}=\frac{v_1}{f}$$

$$\Rightarrow \quad 1+m_1=\frac{v_1}{f} \qquad \ldots (38)$$

where $m_1=\frac{v_1}{u_1}$ = magnification for the first position.

Similarly for the second position of the lens we may write,

$$1+m_2=\frac{v_2}{f} \qquad \ldots (39)$$

From Equation (38) and (39) we get, $m_1-m_2=\frac{v_1-v_2}{f}$

$$\Rightarrow \quad f=\frac{d}{m_1-m_2} \qquad \ldots (40)$$

where $d=v_1-v_2$ is the displacement of the lens.

SOLVED EXAMPLES

A small air bubble in a sphere of glass 3 cm in diameter appears to be 1 cm from the surface when looked along a diameter. Find the true position of the bubble. Given refractive index for glass = 1.5.

ln. We have the general formula for refraction at a curved surface,

$$\frac{n_2}{v}-\frac{n_1}{u}=\frac{n_2-n_1}{r}$$

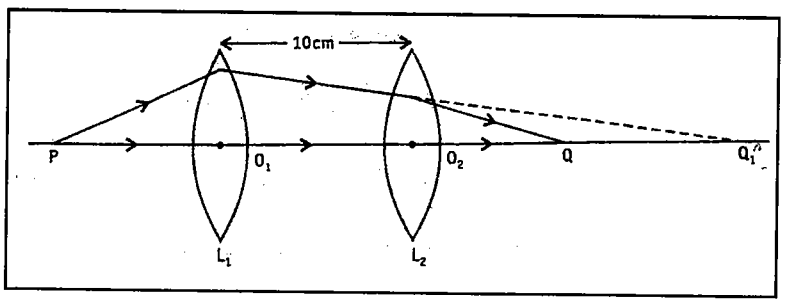
Here refraction takes place through a concave surface and $v=-1$ cm, $r=-1.5$ cm.

$$\Rightarrow \quad \frac{1}{-1}-\frac{1.5}{u}=\frac{1-1.5}{-1.5}$$

or, $u=-1.12$ cm.

Two convex lenses, each of focal length 10 cm, are situated 10 cm apart and have a common axis. An object 2 cm in height is placed on the axis at a distance of 15 cm in front of the first lens. Find the position and size of the final image.

n.



For the first lens (L_1), $u=-O_1P=-15$ cm, $f=10$ cm.

Therefore, image distance $v_1 = O_1Q_1$ is given by $\frac{1}{v_1} - \frac{1}{-15} = \frac{1}{10}$

or, $v_1 = 30$ cm

For the second lens (L_2), object distance, $u_1 = O_2Q_1 = O_1Q_1 - O_1O_2 = 30$ cm $- 10$ cm $= 20$ cm.

Therefore, image distance v for the second lens is given by

$$\frac{1}{\upsilon} - \frac{1}{20} = \frac{1}{10} \quad \Rightarrow \quad v = \frac{20}{3} \text{ cm} = 6\frac{2}{3} \text{ cm}$$

Now $\frac{\text{image height}}{\text{object height}} = \text{total magnification} = \frac{v_1}{u} \times \frac{v}{u_1}$

Therefore, $\frac{\text{image height}}{2} = \frac{30}{-15} \times \frac{20/3}{20} \Rightarrow \text{image height} = -1\frac{1}{3}$ cm

Thus the final image is inverted and of length $1\frac{1}{3}$ cm. It is formed at a distance of $6\frac{2}{3}$ cm from the second lens.

3. Two thin convex lenses of focal lengths 20 cm and 25 cm are placed coaxially 10 cm apart. Find the equivalent focal length.

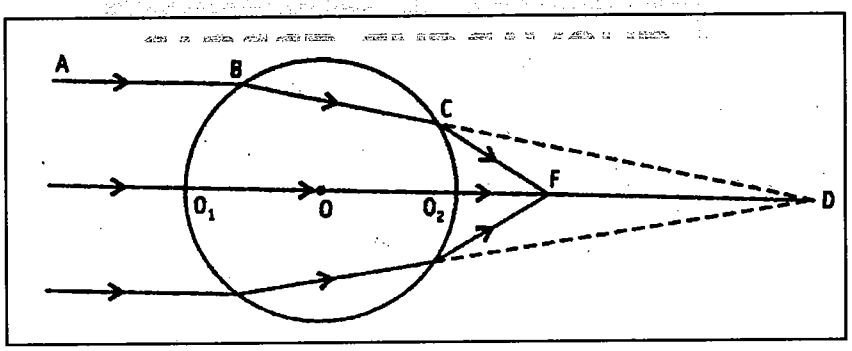
Soln. Equivalent focal length is given by, $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$

Here $f_1 = 20$ cm, $f_2 = 25$ cm, d $= 10$ cm

Therefore, $f = \frac{f_1 f_2}{f_1 + f_2 - d} = \frac{20 \times 25}{20 + 25 - 10}$ cm $= 14\frac{2}{7}$ cm.

4. Consider a transparent sphere of radius 15 cm and of refractive index $n = 1.5$. Find the position of the paraxial focal point.

Soln.



For refraction at the first surface, $\frac{n}{v'} - \frac{1}{u} = \frac{n-1}{r}$

Here $u = -\infty$, r $= +15$ cm, $n = 1.5$; therefore, image distance $v' = O_1D = 45$ cm

For refraction at the second surface D is the virtual object. So object distance $= O_2D$

$= O_1D - O_1O_2 = 45$ cm $- 30$ cm $= 15$ cm

Radius of curvature of the second surface $= -15$ cm

Therefore, in the relation, $\frac{n_2}{v} - \frac{n_1}{u} = \frac{n_2 - n_1}{r}$

we are to put $n_2 = 1$, $n_1 = 1.5$, $u = 15$ cm, $r = -15$ cm.

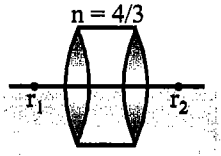
Thus we get $v = +7.5$ cm. So the distance of the paraxial focal point F from the point O_2 is $O_2F = v = +7.5$ cm.

Two thin double convex lenses of refractive index 3/2 are placed in contanct and the space between them is filled with water of refractive index 4/3. What is the focal length of the combination? Assume radius of curvature of each surface = 24 cm.

ln. Focal length of each convex lens is given by $\frac{1}{f_1} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

Here $n = 3/2$, $r_1 = +24$ cm, $r_2 = -24$ cm. $\Rightarrow \frac{1}{f_1} = \left(\frac{3}{2} - 1\right)\left(\frac{1}{24} + \frac{1}{24}\right)$

or, $f_1 = 24$ cm



For the concave lens formed by the water, trapped in double convex lens $n = \frac{4}{3}$, $r_1 = -24\,\text{cm}$, $r_2 = +24\,\text{cm}$.

Focal length of the double concave water lens is given by $\frac{1}{f_2} = \left(\frac{4}{3} - 1\right)\left(-\frac{1}{24} - \frac{1}{24}\right)$

or, $f_2 = -36$ cm

Therefore, equivalent focal length F is given by $\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} = \frac{1}{24} - \frac{1}{36} + \frac{1}{24}$

Therefore, $f = +18$ cm.

When a light ray enters a glass slab from air
(a) its wavelength decreases
(b) its wavelength increases
(c) its frequency increases
(d) neither its wavelenght nor its frequency changes

CAREER ENDEAVOUR

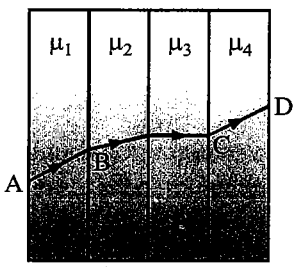
[IIT-JEE : 1980]

ln. (a)

$\lambda = \frac{v}{f}$

In moving from air to glass, f remains unchanged while v decreases. Hence, λ should decrease.

A ray of light passes through four transparent indices μ_1, μ_2, μ_3 and μ_4 as shown in the figure. The surfaces of all media are parallel. If the emergent ray CD is parallel to the incident ray AB, we must have



(a) $\mu_1 = \mu_2$ (b) $\mu_2 = \mu_3$ (c) $\mu_3 = \mu_4$ (d) $\mu_4 = \mu_1$

[IIT-JEE : 2001]

Soln. **(d)**

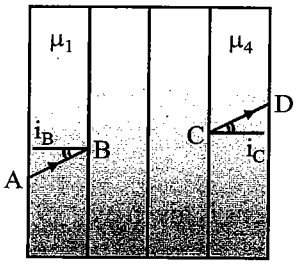
Applying Snell's law at B and C.

Product of μ and i = constant

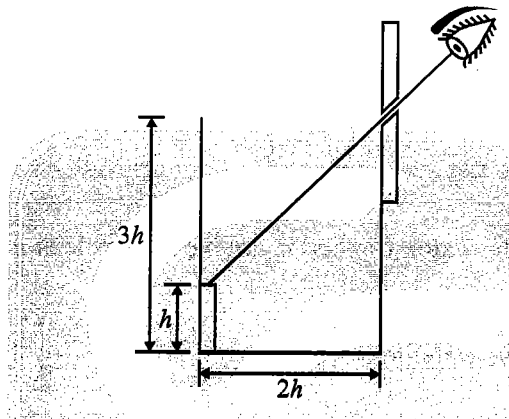
or $\mu_1 \sin i_B = \mu_4 \sin i_C$

But $AB \parallel CD$

$\therefore \; i_B = i_C$ or $\mu_1 = \mu_4$



8. An observer can see through a pin-hole the top end of a thin rod of height h, placed as shown in the figure. The beaker height is $3h$ and its radius h. When the beaker is filled with a liquid up to a height $2h$, he can see the lower end of the rod. Then the refractive index of the liquid is



(a) $\frac{5}{2}$ (b) $\sqrt{\frac{5}{2}}$ (c) $\sqrt{\frac{3}{2}}$ (d) $\frac{3}{2}$

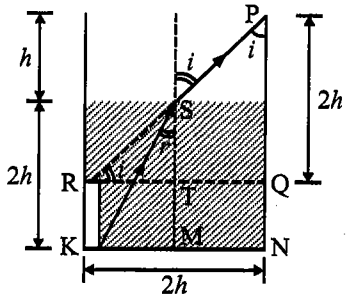
[IIT-JEE : 2002]

Soln. **(b)**

$PQ = QR = 2h \;\therefore\; \angle i = 45°$

$\therefore \; ST = RT = h = KM = MN$ So, $KS = \sqrt{h^2 + (2h)^2} = h\sqrt{5}$

$\therefore \; \sin r = \frac{h}{h\sqrt{5}} = \frac{1}{\sqrt{5}}$



$\therefore \; \mu = \frac{\sin i}{\sin r} = \frac{\sin 45°}{1/\sqrt{5}} = \sqrt{\frac{5}{2}}$

9. A combination of two thin convex lenses of equal focal lengths, is kept separated along the optic axes by a distance of 20 cm between them. The combination behaves as a lens system of infinite focal length. If an object is kept at 10 cm from the first lens, its image will be formed on the other side at a distance x from the second

lens. The value of x is:

(a) 10 cm (b) 20 cm (c) 6.67 cm (d) Infinite.

[IIT-JAM : 2005]

In. (a) $\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1 f_2}$

where d = the separation of two lenses.

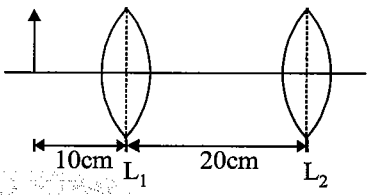
f_1 and f_2 are the focal length of lenses.

F = the focal length of combination system.

Given : $F=\infty,\ f_1=f_2=f$

$$\Rightarrow \quad \frac{1}{\infty}=\frac{1}{f}+\frac{1}{f}-\frac{d}{f^2}$$

$$\Rightarrow \quad 0=\frac{2}{f}-\frac{d}{f^2} \quad \Rightarrow f=\frac{d}{2} \quad \Rightarrow f=\frac{20}{2}=10\ cm$$



For first lens,

$v=?$

$u=-10\ cm$ $\Rightarrow \frac{1}{v}-\frac{1}{u}=\frac{1}{f} \Rightarrow \frac{1}{v}+\frac{1}{10}=\frac{1}{10} \Rightarrow \frac{1}{v}=0 \Rightarrow v=\infty$

$f=10\ cm$

Image formed by the first lens will acts as a virtual object for second lens. Therefore,

$u=\infty$ for second lens

$$\Rightarrow \frac{1}{v}-\frac{1}{\infty}=\frac{1}{f'} \Rightarrow v=f' \Rightarrow v=10\ \text{cm}$$

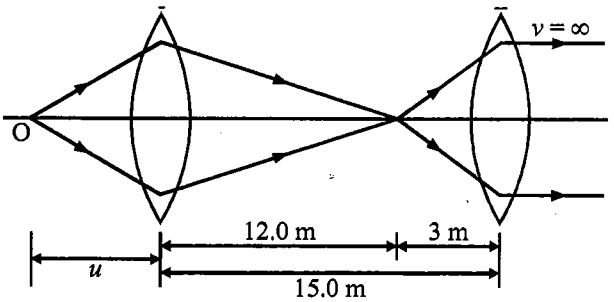
The focal lengths of the objective and the eyepiece of a compound microscope are 2.0 cm and 3.0 cm respectively. The distance between the objective and the eyepiece is 15.0 cm. The final image formed by the eyepiece is at infinity. The two lenses are thin. The distance in cm of the object and the image produced by the objective, measured from the objective lens, are respectively

(a) 2.4 and 12.0 (b) 2.4 and 15.0 (c) 2.0 and 12.0 (d) 2.0 and 3.0

[IIT-JEE : 1995]

n. (a)

Since, the final image is formed at infinity, the image formed by the objective will be at the focal point of the eyepiece, which is 3.0 cm. The image formed by the objective will be at a distance of 12.0 cm (= 15.0 cm – 3.0 cm) from the objective

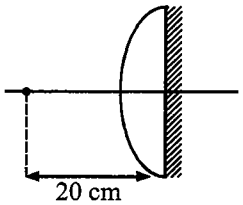


If u is the distance of the object from the objective, we have $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$

$$\Rightarrow \quad \frac{1}{u}+\frac{1}{12.0}=\frac{1}{2.0}$$

$$\Rightarrow \quad u=\frac{(12.0)\,(2.0)}{12.0-2.0}=\frac{24.0}{10.0}=2.4 \text{ cm}$$

11. A point object is placed at a distance of 20 cm from a thin planoconvex lens of focal length 15 cm. The plane surface of the lens is now silvered. The image created by the system is at



(a) 60 cm to the left of the system
(b) 60 cm to the right of the system
(c) 12 cm to the left of the system
(d) 12 cm to the right of the system

[IIT-JEE : 2006]

Soln. **(c)**

Refraction from lens : $\frac{1}{v_1}-\frac{1}{-20}=\frac{1}{15} \Rightarrow v_1=\frac{15\times 20}{20-15}=+60$

i.e., first image is formed at 60 cm to the right of lens system.

Reflection from mirror

After reflection from the mirror, the second image will be formed at equal distance but in opposite side i.e., at distance of 60 cm to the left of lens system.

Refraction from lens

Now, image form by the mirror will act as a object for the lens, thus $u=+60$ cm.

$\frac{1}{v_3}-\frac{1}{60}=\frac{1}{15}$ positive direction **or** $v_3=12$ cm

Therefore, the final image is formed at 12 cm to the left of the lens system.

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Chapter 8

Thick Lens And Lens Combinations

8.1 INTRODUCTION:

There are three pairs of points in an optical system which greatly simplify the description of image formation by the system. These are called cardinal points or Gauss points. For a thick lens (or a system of coaxial surfaces) we have six cardinal points: (a) two principal or unit points, (b) two nodal points and (c) two focal points.

8.1.1 Principal points:

Principal points are a pair of conjugate points on the principal axis of a thick lens, so that if an object be placed at any one of these two points an image of the same size as the object, will be formed at the other point and the image will be on the same side of the principal axis as the object. Thus lateral magnification of the image is positive unity. If planes be drawn through the two principal points perpendicular to the principal axis of the lens then these two planes will be called principal planes or unit planes. The principal planes are, therefore, conjugate planes of unit positive lateral magnification.

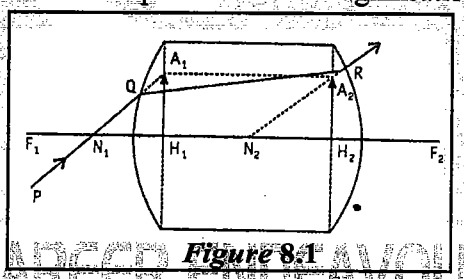

Figure 8.1

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8.1.2 Nodal points:

Nodal points of a thick lens are also a pair of conjugate points on its principal axis so that if a ray be allowed to pass through or travel towards the first point (N_1) then its corresponding emergent ray from the lens will be parallel to the incident ray and will appear to diverge from or pass through the second point (N_2). If planes be drawn through the two nodal points perpendicular to the principal axis of the lens then these two planes will be called nodal planes. As the incident and emergent rays are parallel, they will make equal angles with the principal axis of the lens and hence we may say that nodal planes are conjugate planes having unit angular magnification.

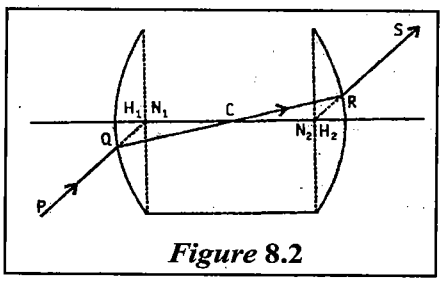

Figure 8.2

8.1.3 Focal Points:

(i) First focal point (F_1): first focal point (F_1) is the position of the object (real or virtual) whose image would be formed at infinity. The distance of first focal point (F_1) from the first principal point H_1 is called first focal length f'.

(ii) Second focal point (F_2) : the second focal point (F_2) is the position of the image (real or virtual) whose object is at infinity. The distance of the second focal point (F_2) from the second principal point H_2 is called second focal length $f (= H_2F_2)$.

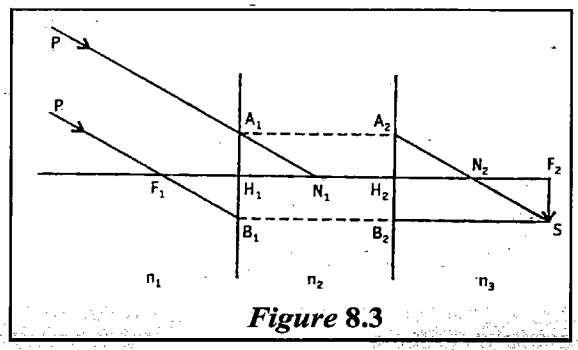

Figure 8.3

8.1.4 Relation between the distances of cardinal points:

The distance (H_1N_1) between the first principal and nodal points = distance (H_2N_2) between the second principal and nodal points.

First focal length $f' (= H_1F_1)$ = distance (F_2N_2) between the second focal and nodal points.

Second focal length $f (= H_2F_2)$ = distance (F_1N_1) between the first focal and nodal points.

The distance between two nodal points (N_1N_2) = distance between two principal points (H_1H_2).

8.2 CARDINAL POINTS OF A THICK LENS:

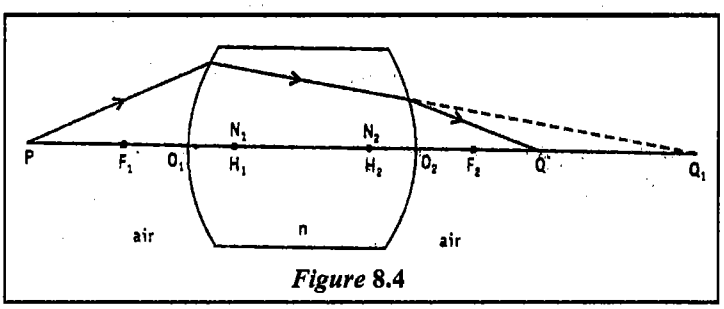

Figure 8.4

Consider a thick lens of thickness $t = O_1O_2$ placed in air. Let P be a point object and Q_1 be its image formed by refraction at the first surface of the lens whose radius of curvature is r_1. This image Q_1 serves as an object to the second surface of radius of curvature r_2 and forms the final image at Q. Let

$$u = -O_1P,\ \upsilon_1 = O_1Q_1,$$

$$O_2Q_1 = O_1Q_1 - O_1O_2 = \upsilon_1 - t \text{ and } \upsilon = O_2Q$$

Now by using the formula for refraction at a single spherical surface we get,

$$\frac{n}{\upsilon_1} - \frac{1}{u} = \frac{n-1}{r_1} \qquad \ldots (1)$$

$$\frac{1}{\upsilon} - \frac{n}{\upsilon_1 - t} = \frac{1-n}{r_2} \qquad \ldots (2)$$

where n is the refractive index of the material of the lens.

Putting $y_1 = \upsilon_1 / n$ and $t' = t/n$ we can rewrite the Equation (1) and (2) as

$$\frac{1}{y_1} - \frac{1}{u} = P_1; \frac{1}{\upsilon} - \frac{1}{y_1 - t'} = P_2 \qquad \ldots (3)$$

where $P_1 = \frac{n-1}{r_1}$ and $P_2 = \frac{1-n}{r_2}$ are respectively the refracting powers of the 1st and 2nd surfaces. Eliminating y_1 from Equation (3), we get, $\upsilon = \frac{cu + d}{au + b}$ ... (4)

where $a = P_1 + P_2 - P_1P_2t', b = 1 - P_2t', c = 1 - P_1t'$ and $d = -t'$

The constants a, b, c and d are interrelated by the relation bc – ad = 1 ... (5)

Note that the Equation (5) is a relation between conjugate distances u and υ measured from the poles O_1 and O_2 of the 1st and 2nd surfaces of the lens respectively.

8.2.1 Principal foci (F_1 and F_2) with reference to poles:

We know that when $\upsilon = \infty, u = -O_1F_1$ = 1st focal length of the lens from O_1; but when $u = \infty, \upsilon = O_2F_2$ = 2nd focal length of the lens from O_2.

Applying these two conditions in Equation (4) we get, $au + b = 0$

$$\Rightarrow \quad u = -O_1F_1 = -b/a \text{ and } \upsilon = O_2F_2 = \frac{c + d/\infty}{a + b/\infty} = \frac{c}{a}$$

Thus, when distances are measured from the poles,

First focal length $(\beta') = -O_1F_1 = -\frac{b}{a}$ and second focal length $(\beta) = O_2F_2 = \frac{c}{a}$

8.2.2 Linear magnification:

Linear magnification at the 1st and 2nd surfaces are given by

$$m_1 = \frac{\upsilon_1 / n}{u} \text{ and } m_2 = \frac{\upsilon}{(\upsilon_1 - t)/n} = \frac{\upsilon}{y_1 - t'}$$

Hence total magnification, $m = m_1m_2 = \frac{y_1}{u} \cdot \frac{\upsilon}{y_1 - t'}$ ... (6)

Using Equation (3) we get,

$$\frac{1}{m} = \frac{u}{y_1} \cdot \frac{y_1 - t'}{\upsilon} = \frac{u}{y_1}\left[1 + P_2(y_1 - t')\right] = \frac{u}{y_1}(1 - P_2t') + P_2u$$

$$= (1 + P_1u)(1 - P_2t') + P_2u = au + b$$

Now from Equation(4),

$$\upsilon(au + b) - cu - d = 0 \quad \Rightarrow \quad a\upsilon(au + b) - acu - ad = 0$$

$$\Rightarrow \quad a\upsilon(au + b) - acu + 1 - bc = 0 \quad \Rightarrow \quad a\upsilon(au + b) - c(au + b) = -1$$

$$\Rightarrow \quad (au+b)(a\upsilon - c) = -1$$

$$\Rightarrow \quad \frac{1}{m} = au + b = \frac{1}{c - a\upsilon} \qquad \ldots (7)$$

8.2.3 Principal or unit points (H_1 and H_2):

By definition principal points or unit points are conjugate points of unit positive linear magnification i.e., $m = 1$. Therefore, the distance (p') of 1st unit H_1 from O_1 i.e., O_1H_1 will u for which $m = 1$

From Equation (7) $au + b = 1$

$$\Rightarrow \quad p' = u = O_1H_1 = \frac{1-b}{a} \qquad \ldots (8)$$

The distance (p) of 2nd principal point H_2 from O_2 is given by υ for which $m = 1$. Therefore, from Equation(7)

$$c - a\upsilon = 1; \text{ or } p = \upsilon = -O_2H_2 = \frac{c-1}{a} \qquad \ldots (9)$$

8.2.4 Nodal points (N_1 and N_2):

The nodal points N_1 and N_2 coincide with the principal points H_1 and H_2 as the media on two sides of the lens are same (air).

8.2.5 Focal lengths (measured from principal points):

The distance of 1st focal point F_1 from the 1st principal point H_1 is the 1st focal length (f') of the thick lens. Thus

$$f' = -H_1f_1 = -(O_1H_1 + O_1F_1) = -\left[\frac{1-b}{a} + \frac{b}{a}\right] \qquad \ldots (10)$$

The distance of 2nd focal point F_2 from the 2nd principal point H_2 is the 2nd focal length (f) of the thick lens. Thus,

$$f = H_2F_2 = O_2H_2 + O_2F_2 = \frac{c-1}{a} + \frac{a}{a} \qquad \ldots (11)$$

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$$\Rightarrow \quad \frac{1}{f} = a = P_1 + P_2 - P_1P_2t' = \frac{n-1}{r_1} + \frac{1-n}{r_2} + \frac{(n-1)^2}{r_1r_2}\cdot\frac{t}{n}$$

$$= (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{(n-1)^2.t}{r_1r_2n} \qquad \ldots (12)$$

In terms of second focal length $f = \frac{1}{a}$,

Equations (8) and (9) takes the following simple forms: $p' = fP_2t'$ and $p = fP_1t'$

8.2.6 Conjugate foci relation with distances measured from H_1 and H_2:

Let x = object distance from 1st principal point $H_1 = -H_1P = -[O_1H_1 + O_1P]$

$$= u - \frac{1-b}{a} \qquad \ldots (13)$$

and y = image distance from 2nd principal point H_2

$$= H_2Q = H_2O_2 + O_2Q = \upsilon - \frac{c-1}{a} \quad \ldots (14)$$

$$\Rightarrow \quad \frac{1}{y} - \frac{1}{x} = a\left[\frac{1}{a\upsilon - c + 1} - \frac{1}{au + b - 1}\right] \quad \ldots (15)$$

Using Equation (4), $\frac{1}{a\upsilon - c + 1} = \frac{1}{a.\frac{cu + d}{au + b} - c + 1} = \frac{au + b}{au + b - 1} \ldots (16)$

Substituting this in the equation (15) we get,

$$\frac{1}{y} - \frac{1}{x} = a = \frac{1}{f} \quad \ldots (17)$$

This equation is identical with the conjugate foci relation of thin lens. Thus we conclude that when the distances of conjugate points of a thick lens are measured from principal points, its formula becomes identical to that of a thin lens.

8.2.7 Conjugate foci relation with distance measured from F_1 and F_2:

Let the object distance measured from 1st focus F_1

$$U = -PF_1 = -(O_1P - O_1F_1) = u + \frac{b}{a}$$

Image distance measured from 2nd focus F_2,

$$V = F_2Q = O_2Q - O_2F_2 = \upsilon - \frac{c}{a}$$

$$\therefore \quad UV = \left(u + \frac{b}{a}\right)\left(\upsilon - \frac{c}{a}\right) = \frac{1}{a^2}(au + b)(a\upsilon - c) = -\frac{1}{a^2} = ff' \quad \ldots (18)$$

This is known as Newton's equation.

8.3 MAGNIFICATIONS PRODUCED BY A THICK LENS:

8.3.1 Linear magnification (m):

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When the distances of conjugate points are measured from principal points of a thick lens, the thick lens formula becomes identical to thin lens formula. Hence linear magnification produced by an thick lens air is,

$$m = \frac{y}{x} \quad \ldots (19)$$

The Equation (17) gives $\frac{1}{x} = \frac{f - y}{yf}$; and $y = -\frac{xf}{x + f}$

$$m = y \times \frac{f - y}{yf} = -\frac{y - f}{f} = -\frac{V}{f} \quad \ldots (20)$$

Also, $$m = \frac{xf}{a + f} \times \frac{1}{x} = \frac{f}{x + f} = \frac{f}{x - f'}$$

$$= \frac{f}{-PH_1 + F_1H_1} = \frac{f}{-PF_1} = \frac{f}{U} \quad \ldots (21)$$

8.3.2 Longitudinal magnification (l):

If δV be the length of the image of an object of length δU, both measured along the axis, then longitudinal magnification of the image is given by,

$$l = \lim \frac{\delta V}{\delta U} = \frac{dV}{dU} \qquad \ldots (22)$$

Differentiating the relation $UV = ff'$ we get,

$$V + U.\frac{dV}{dU} = 0 \qquad \Rightarrow l = \frac{dV}{dU} = -\frac{V}{U}$$

Now $$m^2 = \frac{V}{f} \times \frac{f}{U} = -\frac{V}{U} = l \qquad \ldots (23)$$

Thus longitudinal magnification (l) is equal to the square of the linear magnification.

8.3.3 Angular magnification (M):

If the incident ray and its corresponding emergent ray from the lens make angles α_1 and α_2 with the axis of the lens then angular magnification of the image is given by

$$M = \frac{\tan \alpha_2}{\tan \alpha_1} \qquad \ldots (24)$$

By Helmholtz's law we get, $n_1 y_1 \tan \alpha_1 = n_2 y_2 \tan \alpha_2$.

Here $n_1 = n_2 = 1$ and hence angular magnification

$$M = \frac{\tan \alpha_2}{\tan \alpha_1} = \frac{y_1}{y_2} = \frac{1}{m} \qquad \ldots (25)$$

Thus angular magnification is equal to the reciprocal of its linear magnification (m).

8.3.4 Relation between m, l and M:

From the relations (22) and (24) we get

$$Ml = \frac{1}{m} \times m^2 = m \qquad \Rightarrow \frac{Ml}{m} = 1 \qquad \ldots (26)$$

SOLVED EXAMPLES

1. Find the positions of cardinal points of a thick double convex lens in air whose surfaces have radii of curvature 5 cm and 4 cm, the thickness of the lens being 3 cm and refractive index $n = 1.5$

Soln. Power of the 1st surface $P_1 = \frac{n-1}{r_1} = \frac{1.5-1}{5} = \frac{1}{10}$ cm^{-1}

Power of the 2nd surface $P_2 = \frac{n-1}{r_2} = \frac{1-1.5}{-4} = \frac{1}{8}$ cm^{-1}

$$a = P_1 + P_2 - p_1 P_2 t' = \frac{1}{10} + \frac{1}{8} - \frac{1}{10} \times \frac{1}{8} \times \frac{3}{1.5} = 0.2 \text{ cm}^{-1}$$

$$b = 1 - P_2 t' = 1 - \frac{1}{8} \times \frac{3}{1.5} = 0.75$$

$$c = 1 - P_2 t' = 1 - \frac{1}{10} \times \frac{3}{1.5} = 0.8$$

Locations of the principal points (H_1, H_2) and focal points (F_1, F_2) are given as,

$$p' = O_1H_1 = \frac{1-b}{a} = \frac{1-0.75}{0.2} = +1.25 \text{ cm}$$

$$p = -O_2H_2 = \frac{c-1}{a} = \frac{0.8-1}{0.2} = -1 \text{ cm}$$

$$\beta' = -O_1F_1 = -\frac{b}{a} = -\frac{0.75}{0.2} = -3.75 \text{ cm}$$

$$\beta = O_2F_2 = \frac{c}{a} = \frac{0.8}{0.2} = +4 \text{ cm}$$

$$f' = -H_1F_1 = -\frac{1}{a} = -\frac{1}{0.2} = -5 \text{ cm}$$

$$f = H_2F_2 = \frac{1}{a} = +5 \text{ cm}$$

As the lens is surrounded by air nodal points coincide with principal points.

2. Find the cardinal points of a transparent glass sphere ($n = 1.5$) of radius 1 cm placed in air. Where will sunlight be focussed by such a sphere?

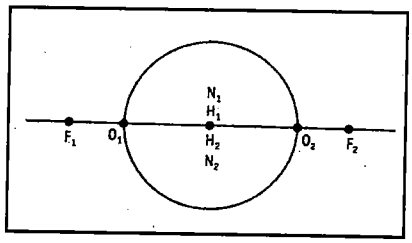
Soln. Power of 1st surface $P_1 = \frac{n-1}{r} = \frac{1.5-1}{1} = \frac{1}{2}$ cm^{-1}

Power of surface $P_2 = \frac{1-n}{-r} = \frac{1-1.5}{-1} = \frac{1}{2}$ cm^{-1}

Thickness $t = 2$ cm, reduced thickness $t' = 2/1.5$ cm

$$\Rightarrow \quad a = P_1 + P_2 - P_1P_2t' = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} \times \frac{2}{1.5} = \frac{2}{3} \text{ cm}^{-1}$$

$$b = 1 - P_2t' = 1 - \frac{1}{2} \times \frac{2}{1.5} = \frac{1}{3}, \; c = \frac{1}{3}$$



Therefore, positions of principal points (H_1, H_2) and focal points (F_1, F_2) are given as,

$$p' = O_1H_1 = \frac{1-b}{a} = \frac{1-\frac{1}{3}}{\frac{2}{3}} = +1 \text{ cm} \quad \text{and} \quad p = -O_2H_2 = \frac{c-1}{a} = \frac{\frac{1}{3}-1}{\frac{2}{3}} = -1 \text{ cm}$$

$$\beta' = -O_1F_1 = -\frac{b}{a} = -\frac{\frac{1}{3}}{\frac{2}{3}} = -0.5 \text{ cm and } \beta = O_2F_2 = \frac{c}{a} = \frac{\frac{1}{3}}{\frac{2}{3}} = +0.5 \text{ cm}$$

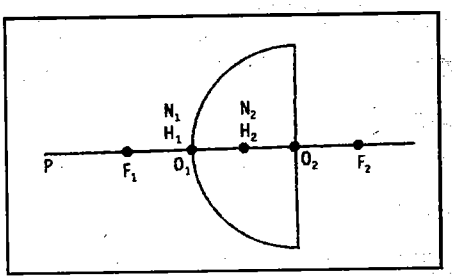
$$f' = -H_1F_1 = -\frac{1}{a} = -\frac{3}{2} = -1.5 \text{ cm and } f = H_2F_2 = \frac{1}{a} = \frac{3}{2} = +1.5 \text{ cm}$$

3. Consider a hemispherical lens of radius of curvature 12 cm and refractive index 2. A small object is placed 36 cm from its curved surface. Find the cardinal points and the position and nature of the image.

Soln. Here $r_1 = r = 12$ cm, $r_2 = \infty$, $P_1 = \frac{n-1}{r} = \frac{2-1}{12} = \frac{1}{12}$ cm^{-1}, $P_2 = 0$

$$t' = \frac{t}{n} = \frac{r}{n} = \frac{12}{2} = 6 \text{ cm.}$$

$$\Rightarrow \quad a = P_1 + P_2 - P_1P_2t' = \frac{1}{12} \text{ cm}^{-1}, \; b = 1, \; c = 1 - P_1t' = 1 - \frac{1}{12} \times 6 = \frac{1}{2}$$



Now, $p' = O_1H_1 = \frac{1-b}{a} = 0$, $p = -O_2H_2 = \frac{c-1}{a} = -6$ cm

$$\beta' = -O_1F_1 = -\frac{b}{a} = -12 \text{ cm}, \; \beta = O_2F_2 = \frac{c}{a} = +6 \text{ cm}$$

$$f' = H_1F_1 = -\frac{1}{a} = -12 \text{ cm}, \; f = H_2F_2 = \frac{1}{a} = +12 \text{ cm}$$

Object distance measured from H_1, $x = -PH_1 = -PO_1 = -36$ cm

Therefore, the image is formed at 18 cm to the right of H_2 in Figure above. The magnification is $\frac{y}{x} = \frac{18}{-36} = -\frac{1}{2}$

4. An object P is placed at a distance of 40 cm in front of a convex lens (O_1) of focal length 20 cm. Behind the convex lens a cancave lens (O_2) of focal length 10 cm is placed at a distance of 15 cm from the convex lens. Find the cardinal points of the lens system and the position of final image from the centre (O_2) of second lens. What is the magnification of final image?

Soln.

$$P_1 = \frac{1}{f_1} = \frac{1}{20} \text{ cm}^{-1}, \; P_2 = \frac{1}{f_2} = -\frac{1}{10} \text{ cm}^{-1}, t = 15 \text{ cm},$$

$$a = P_1 + P_2 - P_1P_2t = \frac{1}{20} - \frac{1}{10} + \frac{1}{20} \times \frac{1}{10} \times 15 = \frac{1}{40}$$

$$b = 1 - P_2t = 1 + \frac{15}{10} = \frac{5}{2}, \; c = 1 - P_1t = 1 - \frac{1}{20} \times 15 = \frac{1}{4}$$

$$\beta' = -O_1F_1 = -\frac{b}{a} = -100 \text{ cm},\ \beta = O_2F_2 = \frac{c}{a} = +10 \text{ cm}$$

$$f' = -H_1F_1 = -\frac{1}{a} = -40 \text{ cm},\ f = H_2F_2 = \frac{1}{a} = +40 \text{ cm},$$

$$p' = O_1H_1 = \frac{1-b}{a} = -60 \text{ cm},\ p = -O_2H_2 = \frac{c-1}{a} = -30 \text{ cm}$$

When the distances are measured from principal points we get,

$$\frac{1}{y} - \frac{1}{x} = \frac{1}{f} \text{ and } m = \frac{y}{x}$$

Here $x = H_1P = -[H_1O_1 + O_1P] = -[-60 + 40] = +20$ cm

$f = H_2F_2 = +40$ cm. Hence y is given by

$$\frac{1}{y} - \frac{1}{20} = \frac{1}{40}; \text{ or, } y = +\frac{40}{3} \text{ cm}$$

If Q is the image position, then $y = H_2Q = \frac{40}{3}$ cm

$$\Rightarrow \quad O_2Q = H_2Q - O_2H_2 = \frac{40}{3} - 30 = -\frac{50}{3} \text{cm}$$

Mabnification $m = \frac{y}{x} = \frac{2}{3}$.

Consider a system of two lenses separated by a distance in air. If p' represents the distance of 1st principal plane from 1st lens and p represents the distance of 2nd principal plane from 2nd lens, show that $p'f_2 + pf_1 = 0$.

oln. $p = O_1H_1 = \frac{1-b}{a}$. Now $\frac{1}{a} = f$ and $b = 1 - P_2t = 1 - \frac{t}{f_2}$

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$$\Rightarrow \quad p' = \frac{f.t}{f_2} \qquad \ldots (1)$$

$$\Rightarrow \quad p = -O_2H_2 = \frac{c-1}{a}.$$

Since $c = 1 - P_1t = 1 - \frac{t}{f_1}$, $p = -\frac{ft}{f_1}$ ... (2)

Combining Equations (1) and (2) we get,

$$p'f_2 + pf_1 = 0$$

Chapter 9

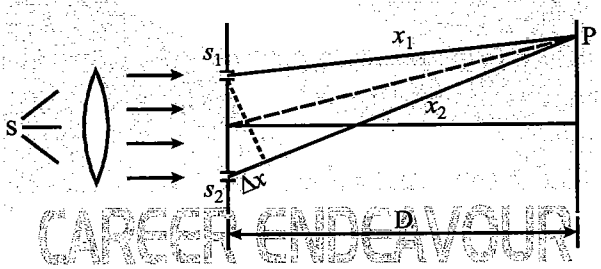
Interference of Light

9.1 INTRODUCTION:

When monochromatic waves of light from two sources proceed almost in the same direction and superpose at a point either in same or in opposite phase. Then the intensity of light at that point will be maximum or minimum according to as the waves meet the point in the same or in opposite phase. This phenomenon is known as interference of light. This phenomenon requires for its explanation that light must have a wave nature.

9.2 THEORY OF INTERFERENCE:

Suppose a narrow slit S is illuminated with monochromatic light of wavelength λ. There are two other slits S_1 and S_2 equidistant from S. According to Huygens' principle cylindrical wavelets spread out from the slit S. As $SS_1 = SS_2$, the wavelets will reach S_1 and S_2 at the same time instant. Hence new secondary wavelets will start from S_1 and S_2 and diverge towards the screen. These wavelets start with equal phase and hence S_1 and S_2 may be considered as coherent sources.



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Let us represent the complex disturbances constituting light waves at S_1 by

$$y = a_1 e^{i\omega t} \qquad \ldots (1)$$

In travelling from S_1 to any point P on the screen the phase of the wave change by $\frac{2\pi}{\lambda} S_1P = \frac{2\pi}{\lambda} x_1 = kx_1$.

Hence at the instant t the complex disturbance at P due to light from S_1 will be

$$y_1 = a_1 e^{i(\omega t - kx_1)} \qquad \ldots (2)$$

Similarly, the disturbance at P due to light from S_2 may be represented by,

$$y_2 = a_2 e^{i(\omega t - kx_2)} \qquad \ldots (3)$$

The resultant disturbance at P, according to the principal of superposition, is given by

$$y = y_1 + y_2 = e^{i\omega t}\left[a_1 e^{-ikx_1} + a_2 e^{-ikx_2}\right]$$

$$= e^{i\omega t}\left[a_1(\cos kx_1 - i\sin kx_1) + a_2(\cos kx_2 - i\sin kx_2)\right]$$

$$= e^{i\omega t}\left[a - ib\right] = e^{i\omega t}\sqrt{a^2 + b^2}.e^{-i\phi};\ \ \phi = \tan^{-1}\frac{b}{a}$$

where $a = a_1 \cos kx_1 + a_2 \cos kx_2$, $b = a_1 \sin kx_1 + a_2 \sin kx_2$

Therefore, $y = Ae^{i(\omega t - \phi)}$ ... (4)

where $A^2 = a^2 + b^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos k(x_2 - x_1)$ ... (5)

Here A is the amplitude of the resultant disturbance. Hence the intensity of light at P is given as

$$I \propto A^2 \quad \Rightarrow \quad I = a_1^2 + a_2^2 + 2a_1a_2 \cos\delta$$

$$= I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2} \cos\delta \quad \text{... (6)}$$

where $\delta = k.(x_2 - x_1) = \frac{2\pi}{\lambda}(x_2 - x_1)$ is the phase difference between the interfering waves. This phase difference depends on the path difference $x_2 - x_1$ and hence on the position of the point P.

Case-I : Maximum intensity (constructive interference)

when $\delta = 2m\pi$, or, $x_2 - x_1 = 2m\frac{\lambda}{2}$; $m = 0, 1, 2, 3, ...,$

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1}\sqrt{I_2} \qquad [\because \cos 2m\pi = 1]$$

$$= \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$$

Thus when the path difference of the point P from the two sources is even multiple of $\lambda / 2$ the intensity of light at that point becomes maximum and we get a bright band there. This is known as constructive interference.

Case-II : Minimum intensity (destructive interference)

when $\delta = (2m+1)\pi$ or $x_2 - x_1 = (2m + 1)\lambda / 2$; $m = 0, 1, 2, 3,....$

$$I_{min} = I_1 + I_2 - 2\sqrt{I_1}\sqrt{I_2} \qquad [\because \cos(2m+1)\pi = -1]$$

$$= \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$$

when the path difference of the point P from the two sources is odd multiple of $\lambda / 2$ the intensity of light at the point is minimum and we get a nearly dark band there. This is known as destructive interference.

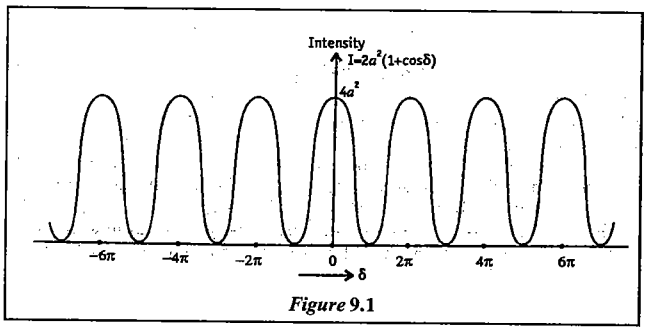

Figure 9.1

(i) No destruction of light energy occurs in interference of light. What happens is merely a redistribution of energy. The energy which disappears from the dark regions actually goes to the bright regions. The average value of energy over a number of fringes can be shown to be the same as if the interference effects were absent. Taking the constant of proportionality to be one, the intensity I can be expressed from Equation .... as

$$I = a_1^2 + a_2^2 + 2a_1a_2 \cos\delta \quad \text{... (7)}$$

where a_1 and a_2 are the amplitudes of the waves and δ is the phase difference.

$$\Rightarrow \quad I_{average} = \frac{\int_0^{2\pi} Id\delta}{\int_0^{2\pi} d\delta} = a_1^2 + a_2^2 = \text{sum of intensities of individual waves.} \quad \ldots (8)$$

Thus we can say that there is no destruction of energy or the principle of conservation of energy is not violated in the phenomenon of interference.

(ii) When light waves from two separated source meet at a point, the point will be bright or dark according to as the waves meet in the same or opposite phases. This phase difference depends on the initial phase difference between the sources and the optical path difference of the point from the two sources. The path difference of the point in question remains constant but as two different source cannot be coherent, the initial phase difference goes on changing very rapidly with time. As a result at every point there is rapid alternation of brightness and darkness and hence we get general illumination.

Ex. 1. Two coherent monochromatic light beams of intensities I and $4I$ are superposed. The maximum and minimum possible intensities in the resulting beam are

(a) $5I$ and I (b) $5I$ and $3I$ (c) $9I$ and I (d) $9I$ and $3I$

[IIT-JEE : 1988]

Soln. **(c)**

$$I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2 = \left(\sqrt{4I} + \sqrt{I}\right)^2 = 9I$$

$$I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2 = \left(\sqrt{4I} - \sqrt{I}\right)^2 = I$$

Ex. 2. Two beams of light having intensities I and $4I$ interfere to produce a fringe pattern on a screen. The phase difference between the beams is $\pi/2$ at point A and π at point B. Then the difference between resultant intensities at A and B is

(a) $2I$ (b) $4I$ (c) $5I$ (d) $7I$

[IIT-JEE : 2001]

Soln. **(b)**

$$I(\phi) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi \quad \ldots(i)$$

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Here, $I_1 = I$ and $I_2 = 4I$

At point $A, \phi = \frac{\pi}{2}$ $\quad \therefore\ I_A = I + 4I = 5I$

At point $B, \phi = \pi$ $\quad \therefore\ I_B = I + 4I - 4I = I$

$\therefore\ I_A - I_B = 4I$

Ex. 3. Two beams of monochromatic light with intensities I_1 and I_2 interfere constructively to produce an intensity of 100 mW. If one of the beams is shifted in phase by 60 degrees, the intensity reduces to 84 mW. Then I_1 and I_2 are

(a) 64 mW and 4 mW (b) 36 mW and 16 mW
(c) 10 mW and 9 mW (d) 92 mW and 8 mW

[IISc : 2011]

Soln. **(a)**

After interference of two beams, the intensity of beams are given by

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$$

For constructive interference, $\delta = 2n\pi$

$100 = I_1 + I_2 + 2\sqrt{I_1 I_2}$ ...(i)

If one of the beam is shifted in phase by 60°, i.e. $\delta = 60°$

$84 = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\pi/3 = I_1 + I_2 + \sqrt{I_1 I_2}$ ...(ii)

By (i) divide (ii),

$\Rightarrow \sqrt{I_1 I_2} = 16$

Putting this in (ii)

$I_1 + I_2 = 84 - \sqrt{I_1 I_2} = 68$

So, $I_1 + I_2 = 68$ and $\sqrt{I_1 I_2} = 16$

Which are satisfied by $I_1 = 64\ mW$ and $I_2 = 4\ mW$

9.3 WIDTH AND SHAPE OF INTERFERENCE FRINGES:

9.3.1 Fringe width:

Let us consider two coherent sources S_1 and S_2 which are sending monochromatic light of wavelength λ (Figure (9.2) The point C on the screen is equidistant from S_1 and S_2 and hence the waves from S_1 and S_2 arrive at C at the same time. If the phase difference between the two coherent sources is zero the waves from S_1 and S_2 will meet at C in same phase and will produce a bright band known as central bright band. Let the mth bright band be formed at P whose distance from C is x_m (say). Now in Figure -9.2, D is the separation between the sources and the screen; d is the separation between the coherent sources.

$$(S_2P)^2 = D^2 + \left(x_m + \frac{d}{2}\right)^2 \Rightarrow S_2P = D\left[1 + \left(\frac{x_m + \frac{d}{2}}{D}\right)^2\right]^{\frac{1}{2}} \quad \ldots (9)$$

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Since usually, $D >> d$, $S_2P \approx D\left[1 + \frac{1}{2}\left(\frac{x_m + \frac{d}{2}}{D}\right)^2\right]$ ... (10)

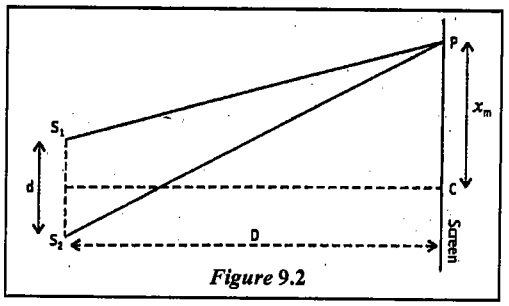

Figure 9.2

Similarly, $S_1P \approx D\left[1+\frac{1}{2}\left(\frac{x_m-\frac{d}{2}}{D}\right)^2\right]$ ... (11)

Now for mth order bright fringe at P,

Path difference $S_2P - S_1P = m\lambda$

Using Equation (10) and (11) we get, $\frac{x_m.d}{D} = m\lambda$

$$x_m = m\frac{\lambda D}{d} \quad \text{... (15)}$$

Similarly, the distance of (m + 1)th bright fringe from C will be

$$x_m = (m+1)\lambda\frac{D}{d} \quad \text{... (16)}$$

Thus the distance between two consecutive bright bands would be

$$\beta = x_{m+1} - x_m = \frac{\lambda D}{d} \quad \text{... (17)}$$

where β is known as fringe width.

If P be the position of mth order dark band then

$$x_m = (2m+1)\frac{\lambda}{2}.\frac{D}{d} \text{ and } x_{m+1} = (2m+2+1)\frac{\lambda}{2}.\frac{D}{d}$$

Hence $\beta = x_{m+1} - x_m = \frac{\lambda D}{d}$ ... (18)

Equations (17) and (18) show that the distances (β) between two consecutive bright or dark bands are equal.

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9.3.2 Shape of interference fringes:

An idea regarding the shape of interference fringes can be obtained by finding the locus of points having a constant path difference from the sources (slits) S_1 and S_2. Referring to Figure -9.3 we choose the mid point O between the slits as the origin of a coordinate system. Let the x-axis be along OX and y-axis be perpendicular to the plane containing the slits. For any point $P(y, x)$ we can write

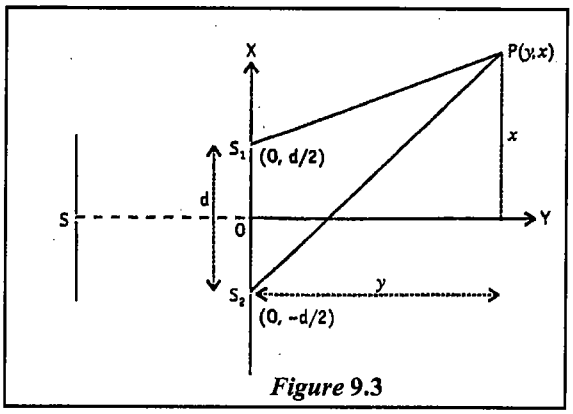

Figure 9.3

$$(S_1p)^2 = y^2 + \left(x - \frac{d}{2}\right)^2$$

and $$(S_2p)^2 = y^2 + \left(x + \frac{d}{2}\right)^2 \quad \ldots (19)$$

where 'd' is the separation between the the slits.

Let $\Delta = S_2P - S_1P$

$$\Rightarrow \quad S_2P = \Delta + S_1P$$

$$\Rightarrow \quad S_2P = \left[y^2 + \left(x + \frac{d}{2}\right)^2\right]^{\frac{1}{2}} = \Delta + \left[y^2 + \left(x - \frac{d}{2}\right)^2\right]^{\frac{1}{2}}$$

On squaring and re-arranging

$$\Rightarrow \quad 2\Delta\left[y^2 + \left(x - \frac{d}{2}\right)^2\right]^{\frac{1}{2}} = 2xd - \Delta^2$$

Again, squaring and re-arranging, we get

$$\Rightarrow \quad \frac{x^2}{\Delta^2/4} - \frac{y^2}{\left(d^2 - \Delta^2\right)/4} = 1 \quad \ldots (20)$$

Thus the loci of points of constant path difference Δ in xy plane are hyperbolae with S_1 and S_2 as foci on x-axis. The eccentricities of the hyperbolae are given by

$$e = \frac{d}{\Delta} \quad \ldots (21)$$

In optical experiments the path difference $\Delta \sim 10^{-8}$ cm and $d \sim 10^{-2}$ cm. Therefore, e is very high and as a consequence the hyperbolae become practically straight lines given by

$$y = \pm\left[\frac{d^2 - \Delta^2}{\Delta^2}\right]^{\frac{1}{2}} x \quad \ldots (22)$$

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If two coherent point source are used the loci of maxima of different orderes will be a system of confrontal hyperboloids.

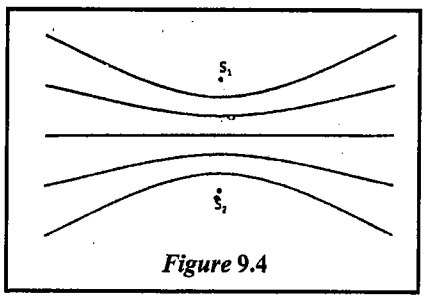
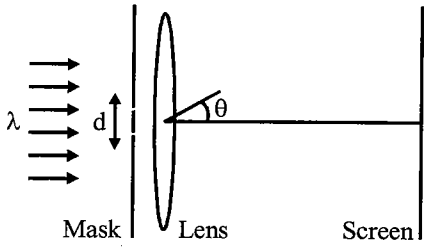


Figure 9.4

ɛ. 4. In an optical telescope arrangement shown below the opaque mask covers the lens completely except for two very small holes, separated by a distance d. The screen is kept at the focal plane of the lens, and a collimated monochromatic radiation with wavelength λ is incident on the mask from left. The angular dependence of the intensity distribution $f(\theta)$ at the screen is given by,



(a) $f(\theta) = 2\pi d \sin\theta/\lambda$

(b) $f(\theta) = 1 + \cos(2\pi d \sin\theta/\lambda)$

(c) $f(\theta)$ = constant

(d) $f(\theta) = 1 - \cos(2\pi d \sin\theta/\lambda)$

[IISc : 2006]

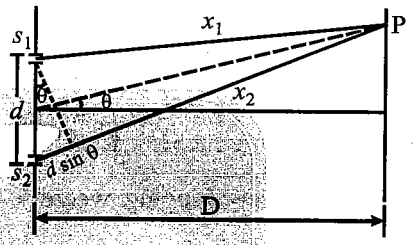
Soln. **(a)** The intensity at any point P on the screen will depend on the plase difference between two beams which interfere at P.

The phase difference

$$\delta = \frac{2\pi}{\lambda} \times \text{path difference}$$

$$= \frac{2\pi}{\lambda} \cdot [S_2P - S_1P]$$

$$= \frac{2\pi}{\lambda} \times d \sin\theta = \frac{2\pi d \sin\theta}{\lambda}$$



So, $f(\theta) = \dfrac{2\pi d \sin\theta}{\lambda}$.

9.4 INTERFERENCE WITH WHITE LIGHT AND COLOUR EFFECT:

The distance of the mth bright fringe from the central one is given by

$$d \sin\theta = d \cdot \frac{x_m}{D} = n\lambda \Rightarrow x_m = m \cdot \frac{\lambda D}{d} \quad \ldots (23)$$

where d is the separation between the two coherent sources and D is the distance of the screen from the source. As x_m is a function of wavelength λ, so the fringes of different colours will be in step only for the central band ($m = 0$). In this case $x_m = 0$ and bright fringe of all wavelengths will coincide and the central band will be white with white light. For higher order bright fringes ($m > 0$) x_m will be greater for a light of longer wavelength and less for a light of shorter wavelength. As the wavelength λ_r for red light is longer than the wavelength λ_v for violet we infer that all bright bands, excepting the central bright band, will be coloured in which red will be in the outermost position while violet will be in the innermost position.

Ex. 5. If the monochromatic source of light in Young's double-slit experiment is replaced by a white-light source, then:

(a) No fringes will be formed **[DU 2014]**

(b) There will be white fringes only

(c) There will be a coloured central fringe only

(d) There will be a central white fringe flanked on either side by a few coloured fringes

Soln. **(d)** This is because at centre there will be maxima and at the either sides, the diffracted rays of different wavelength will interfere.

x. 6. Interference and diffraction of light supports its: **[DU 2014]**

(a) Wave nature (b) Quantum nature

(c) Transverse nature (d) Electromagnetic character

oln. (a)

9.5 CONDITIONS FOR OBSERVABLE INTERFERENCE PATTERN:

(i) The two beams of light which interfere must be coherent.

(ii) The interfering waves must have the same frequency. Also, their amplitudes must be equal or very nearly equal.

(iii) The original source must be monochromatic or very nearly monochromatic. If the light source is heterogeneous the optical path difference between the interfering beams must be kept very small.

(iv) The two interfering beams must propagate along the same direction or must intersect at a very small angle.

(v) For interference with polarised light the waves must be in the same state of polarisation.

x. 7. In a Young's double slit experiment, 12 fringes are observed to be formed in a certain segment of the screen when light of wavelength 600 nm is used. If the wavelength of light is changed to 400 nm, number of fringes observed in the same segment of the screen is given by

(a) 12 (b) 18 (c) 24 (d) 30

[IIT-JEE : 2001]

)ln. (b)

Fringe width, $\beta_1 = \frac{\lambda_1 D}{d}$, $\lambda_1 = 600$ nm and $\beta_2 = \frac{\lambda_2 D}{d}$, $\lambda_2 = 400$ nm

Let n be the number of the fringe then,

$12\beta_1 = n\beta_2$

$\Rightarrow n = 12\frac{\beta_1}{\beta_2} = 12\frac{\lambda_1}{\lambda_2} = 12\frac{600}{400} = 18.$

x. 8. In Young's double slit experiment intensity at a point is (1/4) of the maximum intensity. Angular position of this point is

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(a) $\sin^{-1}\left(\frac{\lambda}{d}\right)$ (b) $\sin^{-1}\left(\frac{\lambda}{2d}\right)$ (c) $\sin^{-1}\left(\frac{\lambda}{3d}\right)$ (d) $\sin^{-1}\left(\frac{\lambda}{4d}\right)$

[IIT-JEE : 2005]

)ln. (c)

$I = I_{max}\cos^2\left(\frac{\phi}{2}\right)$

$\therefore \frac{I_{max}}{4} = I_{max}\cos^2\frac{\phi}{2} \Rightarrow \cos\frac{\phi}{2} = \frac{1}{2}$ or $\frac{\phi}{2} = \frac{\pi}{3} \Rightarrow \phi = \frac{2\pi}{3}$ ...(i)

Also, $\phi = \left(\frac{2\pi}{\lambda}\right)\Delta x$ where $\Delta x = d\sin\theta$

Substituting in equation (i), we get

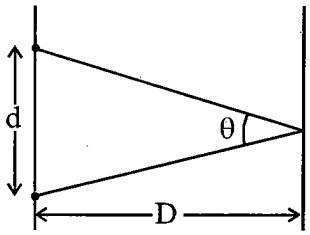
$\sin\theta = \frac{\lambda}{3d}$ or $\theta = \sin^{-1}\left(\frac{\lambda}{3d}\right)$ **Ans.**

Ex. 9. Two coherent plane waves of light of equal amplitude, and each of wavelength $20\pi\times10^{-8}$ m, propagating at an angle of $\frac{\pi}{1080}$ radian with respect to each other, fall almost normally on a screen. The fringe-width (i mm) on the screen is:

(a) 0.108 (b) 0.216 (c) 1.080 (d) 2.160

[IIT-JAM : 2008]

Soln.



From figure, $\tan\theta \simeq \theta = \frac{d}{D}$

Fringe width, $\Delta x = \frac{D\lambda}{d} = \frac{\lambda}{\theta} = \frac{20\pi\times10^{-8}}{\frac{\pi}{1080}} = 21600\times10^{-8}\text{m} = 0.216\text{ mm}$

Ex. 10. A Young's double slit apparatus has a slit separation of 0.5 mm and a screen placed at a distance of 100 cm from the slits. When illuminated by a 632 nm Helium-Neon laser, the position of the first intensity minimum with respect to the central maximum will be **[IISc : 2009, 2012]**

(a) 31.6 μm (b) 63.2 μm (c) 316 μm (d) 632 μm

Soln. (d)

Slit separation $d = 0.5\text{ mm} = 5\times10^{-4}\text{ m}$

Distance between slit and screen $D = 100\text{ cm} = 1\text{ m}$

Wavelength $\lambda = 632\text{ nm} = 632\times10^{-9}\text{ m}$

Position of minima is given by

$$x_n = \left(n+\frac{1}{2}\right)\frac{\lambda}{d} \quad n = 0, 1, 2, \ldots$$

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First intensity minima will correspond to $n = 0$

$$x_0 = \frac{\lambda}{2d} = \frac{632\times10^{-9}\times1}{2\times5\times10^{-4}} = 632\ \mu m$$

Ex. 11. In a Young;s double-slit experiment, the two slits are such that the dark fringes are perfectly dark (i.e., the intensity vanishes at these points). Now, we double the width of one of the two slits. What will be the effect on the interference pattern?

(a) The fringe pattern as well as contrast between the dark and bright fringes will change

(b) The fringe pattern will not change but the contrast between the dark and bright fringes will change.

(c) The fringe pattern will change but the contrast between the dark and bright fringes will not change.

(d) There will be no change in either the interference pattern or the contrast between the dark and bright fringes.

[JNU : 2009]

Soln. Width of the slit are proportional to the amplitude of interfering waves when both slits are same, amplitudes are same, i.e.,

$a_1 = a_2 = a$ (say)

$\Rightarrow I_1 = I_2 = I$ (say)

$\Rightarrow I_{dark} = \sqrt{I_1} - \sqrt{I_2} = 0$

when one slits are double,

$a_1 = a \Rightarrow I_1 = I$ and $a_2 = 2a \Rightarrow I_2 = 4I \left[I \propto a^2\right]$ then,

$I_{dark} = \sqrt{I_1} - \sqrt{I_2} = \sqrt{I} - 2\sqrt{I} \neq 0$

i.e., when width of slits are double, dark bands now are not completely dark. Also white bands willhave more contrast. As fringe pattern depends upon wavelength only, therefore, it will remain same.
Correct option is (b).

9.6 TWO CLASSES OF INTERFERENCE:

(i) Division of wavefront:

Optical devices which divide the incident wavefront into two parts by reflection, refraction or diffraction and thereby give rise to two coherent interfering beams come under the division of wavefront class. In order to maintain spatial coherence it is essential to use narrow sources in these cases. The formation of fringes by **Biprism, Lloyd's single mirror, Billet's divided lens etc.** belong to this category.

(ii) Division of amplitude:

Optical devices which divide the amplitude of incident light wave into two or more parts by partial reflection and refraction and there by give rise to two or more coherent interfering beams of light come under the division of amplitude class. Here we require to use broad source of light. As the interference effects corresponding to different points of the source are superposed here we get brighter bands. Since a large section of the wavefront is used diffraction effects are minimised. The formation of fringes **by thin films, Newton's ring, Michelson's interferometer, Fabry-Perot interferometer etc. belong to this category.**

9.7 FRESNEL'S BIPRISM:

A biprism is essentially two prisms each of very small refracting angle (~ 30′) placed based to base. The incident wavefront is divided into two parts and suffer separate refractions through the two halves of the biprism. The two refracted wavefronts appear to diverge from two virtual sources S_1 and S_2.

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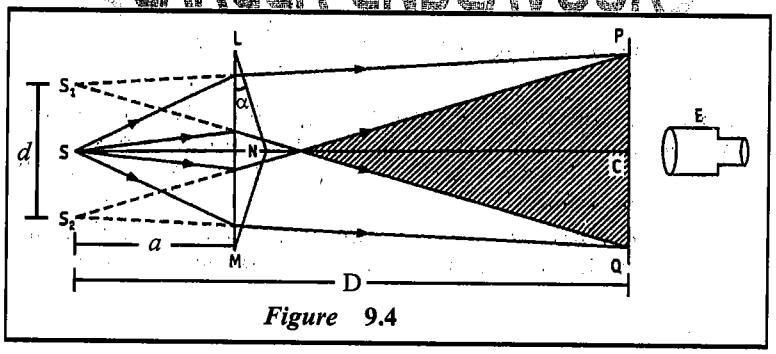


Figure **9.4**

As in young double slit experiment,

The position of bright fringe $x_n = \frac{n\lambda D}{d}$, $n = 0, 1, 2, 3, ...$

The position of dark fringe $x_n = \frac{(2n+1)}{2} \frac{\lambda D}{d}$, $n = 1, 2, 3, ...$

9.7.1 Determination of unknown wavelength:

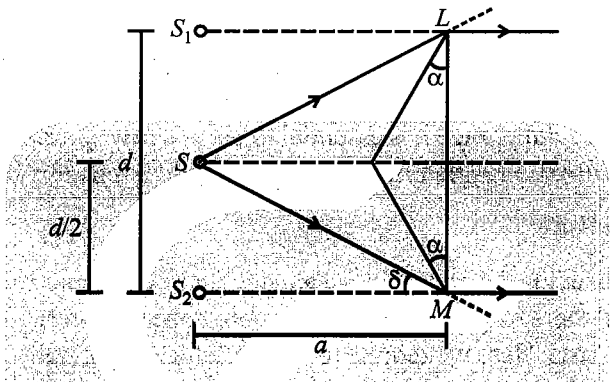
Fringe width β is given by $\beta = \frac{\lambda D}{d}$

where, D = distance between the slit and the screen.
d = distance between the two virtual coherent sources S_1 and S_2.

Unknown wavelength, $\lambda = \frac{d\beta}{D}$ ... (24)

9.7.2 Measurement of the acute angle of biprism:

It is evident that the deviation of each of the rays *SL* and *SM* after their passage through the two thin prisms having acute angles at *L* and *M* respectively, is given by $\delta = \angle SMS_2 = \angle SLS_1$.



Since biprism is thin, deviation produced in a ray is given by

$\delta = (n-1)\alpha$

where *n* is refractive index of the prism.

From figure, $\tan\delta \simeq \delta = \frac{d/2}{a}$

$\Rightarrow d = 2a(n-1)\alpha$ ... (25)

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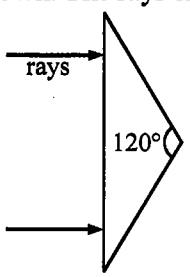
Ex. 12. An isosceles prism of angle 120° has a refractive index 1.44. Two parallel rays of monochromatic light enter the prism parallel to each other in air as shown. The rays emerging from the opposite face



(a) are parallel to each other
(b) are diverging
(c) make an angle $2[\sin^{-1}(0.72) - 30°]$ with each other
(d) make an angle $2\sin^{-1}(0.72)$ with each other

[IIT-JEE : 1995]

Soln. (c)

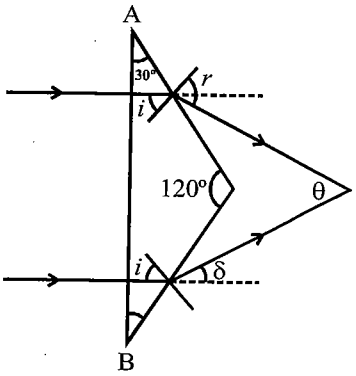
The diagramatic representation of the given problems is shown in figure.

From figure it follows that $\angle i = \angle A = 30°$

From Snell's law, $n_1 \sin i = n_2 \sin r$ **or** $\sin r = \dfrac{1.44 \sin 30°}{1} = 0.72$

Now, $\angle \delta = \angle r - \angle i = \sin^{-1}(0.72) - 30°$

$\therefore \quad \theta = 2(\angle \delta) = 2\{\sin^{-1}(0.72) - 30°\}$



9.7.3 Measurement of the thickness of a thin film:

Let S_1 and S_2 be the two virtual coherent sources which are producing interference fringes on the screen so that C is the position of the central bright band of zero optical path difference i.e. $S_1C = S_2C$.

If a thin film of thickness t be introduced into one of the path (say, S_1P) of the interfering rays, then the position of the central fringe will be shifted from C to P (say), so that the optical path S_1P is again equal to the optical path S_2P. The time taken by light in going from S_1 to P and from S_2 to P will be equal. Thus,

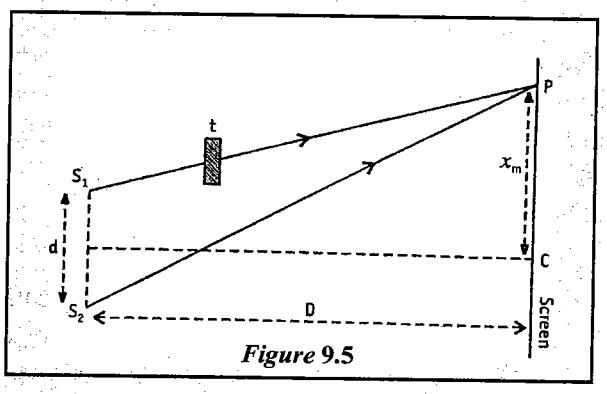


Figure 9.5

$$\frac{S_2P}{c} = \frac{S_1P - t}{c} + \frac{t}{v} \qquad \ldots (26)$$

$$\Rightarrow \qquad S_2P - S_1P = (n-1)t \qquad \ldots (27)$$

where c and v are the velocity of light in air and in the film; and $n = c/v$ is the refractive index of the material of the film, if P is the position originally occupied by the mth order fringe then

$$S_2P - S_1P = m\lambda \qquad \ldots (28)$$

Therefore, from Equation (27) and (28) we get,

$$(n-1)t = m\lambda \qquad \ldots (29)$$

The lateral shift of the central fringe of zero optical path difference is given as

$$x_m = CP = m\beta \qquad \ldots (30)$$

where $\beta = \dfrac{\lambda D}{d}$ is the fringe width

From Equation (29) and (30) we get

$$(n-1)t = \frac{x_m}{\beta}.\lambda \qquad \Rightarrow \quad t = \frac{x_m.\lambda}{\beta(n-1)}$$

$$\Rightarrow \quad t = \frac{x_m}{(n-1)} \cdot \frac{d}{D} \qquad \ldots (31)$$

Ex. 13. In the ideal double-slit experiment, when a glass-plate (refractive index 1.5) of thickness t is introduced in the path of one of the interfering beams (wavelength λ), the intensity at the position where the central maximum occurred previously remains unchanged. The minimum thickness of the glass-plate is

(a) 2λ (b) $\frac{2\lambda}{3}$ (c) $\frac{\lambda}{3}$ (d) λ

[IIT-JEE : 2002]

Soln. **(a)**

Path difference due to slab should be integral multiple of λ i.e. $\Delta x = n\lambda \Rightarrow (\mu - 1)t = n\lambda \quad n = 1, 2, 3, \ldots$

or $t = \frac{n\lambda}{\mu - 1}$.

For minimum value of t, $n = 1$

$$\therefore \quad t = \frac{\lambda}{\mu - 1} = \frac{\lambda}{1.5 - 1} = 2\lambda$$

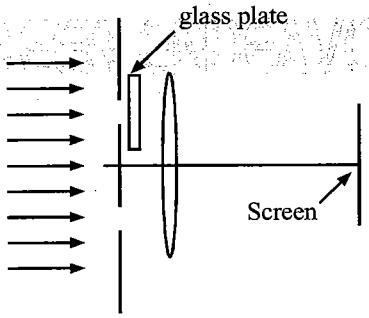
Ex. 14. Young's double slit experimental measures

(a) temporal coherence of light
(b) spatial coherence of light
(c) polarization of light source
(d) beat frequency of light source

[IIT-JAM GP : 2011]

Ans. (b)

Ex. 15. In a Young's double-slit experiment, the central maximum occurs at zero path difference between the interfering beams. A glass plate of thickness 0.005 mm is inserted to cover the upper slit completely (see figure). If the slit separation is 0.1 mm and the diffraction pattern is observed at the focal plane of the converging lens (focal length 200 mm), which of the following statements on the position of the central maximum is true ?
[refractive index of glass is 1.5]



(a) The central maximum shifts up by 15 mm
(b) The position of the central maximum remains unchanged
(c) The central maximum shifts down by 15 mm
(d) The central maximum shifts down by 1.5 mm

[IISc : 2007]

Soln. From equation 31, central maxima will shift by distance

$$x_m = \frac{(\mu - 1)\, t}{d} \cdot D = \frac{0.5 \times 0.005 \times 200}{0.1} = 5 \text{ mm}$$

9.8. LLOYD'S SINGLE MIRROR:

9.8.1 Introduction:

In Lloyd's single mirror arrangement (Figure 9.6) light from a narrow slit S_1 illuminated with a monochromatic light is partly incident at a grazing angle on a metallic mirror M_1M_2 while the rest reaches the screen AB directly. The reflected light appears to diverge from a virtual source S_2. Hence S_1 and S_2 act as coherent sources and interference fringes are formed in the region of overlapping EF.

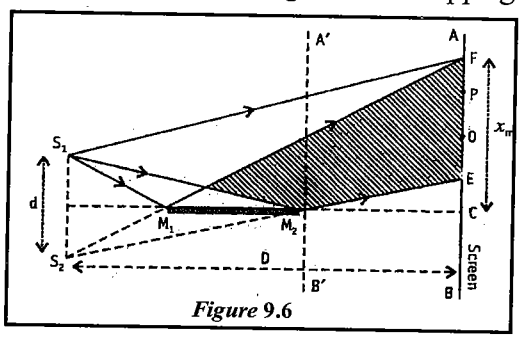


Figure 9.6

Ray reflected by the mirror suffer a phase change of π, therefore, effective path changes by $\lambda/2$.

Condition of maxima

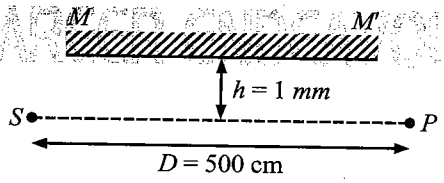
$$\Delta x = d\sin\theta + \frac{\lambda}{2} = n\lambda \Rightarrow d\sin\theta = (2n-1)\frac{\lambda}{2},\ n = 1,2,3,....$$

Condition of minima

$$\Delta x = d\sin\theta + \frac{\lambda}{2} = (2n+1)\frac{\lambda}{2} \Rightarrow d\sin\theta = n\lambda,\ n = 0,1,2,3,...$$

Fringe width is same as in YDSE, i.e., $\beta = \frac{\lambda D}{d}$

ĸ. 16. A simple setup to study interference of light (Lloyd's Mirror) consists of a monochromatic point source S of wavelength $\lambda = 5000$ Å and a detector P separated from it by a distance of D = 500 cm. A plane mirror MM' is placed above at height h which is initially adjusted to be $h = 1$mm so that bright fringe is observed at P.



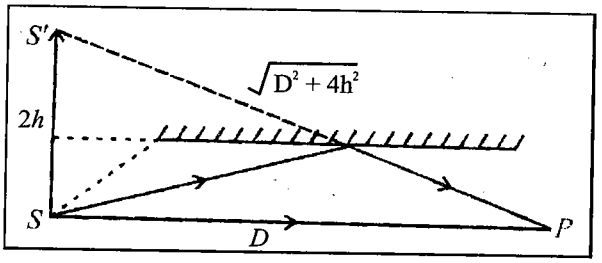
By what distance must the mirror be moved vertically so that the next bright fringe is observed at P?

(a) 25.0 μm (b) 12.5 μm (c) 6.25 μm (d) 3.12 μm

[IISc : 2008]

In. (None)

Since, ray reflected from the mirror suffer a additional path change $\frac{\lambda}{2}$.



Path difference $\Delta x = S'P - SP = \sqrt{D^2 + 4h^2} - D + \frac{\lambda}{2} = n\lambda$.

If this path difference corresponds to nth bright fringe. Let h' be the new height which corresponds to $(n+1)^{th}$ bright fringe

$$\Delta x' = \sqrt{D^2 + 4h'^2} - D + \frac{\lambda}{2} = (n+1)\lambda$$

$$\Delta x' - \Delta x = \sqrt{D^2 + 4h'^2} - \sqrt{D^2 + 4h^2} = \lambda$$

$$\Rightarrow D\left[1 + \frac{4h'^2}{D^2}\right]^{1/2} - D\left[1 + \frac{4h^2}{D^2}\right]^{1/2} = \lambda$$

$$\Rightarrow D\left[1 + \frac{1}{2}\cdot\frac{4h'^2}{D^2}\right] - D\left[1 + \frac{1}{2}\cdot\frac{4h^2}{D^2}\right] = \lambda \quad \Rightarrow \quad \frac{2(h'^2 - h^2)}{D} = \lambda$$

$$h'^2 = \frac{\lambda D}{2} + h^2 = \frac{5\times10^{-7}\times5}{2} + 10^{-6} = 2.25\times10^{-6}$$

$h' = 1.5\times10^{-3} = 1.5\ mm$

So, mirror should be moved vertically be a distance of $1.5 - 1 = 0.5\ mm$.

9.8.2 Difference between fresnel's biprism and Lloyed's mirror fringes:

(i) In biprism experiment fringes are formed on both sides of the central fringe, whereas in Lloyd's mirror arrangement less than half of the fringes are obtained on one side of the central line.

(ii) In biprism the central fringe is bright whereas in Lloyd's mirror it is dark.

(iii) In Fresnel's biprism the separation (d) between every pair of corresponding points of the coherent sources is same and hence fringe width is same for all parts of the source. But in Lloyd's mirror due to lateral inversion in the mirror d is different for different pair of coherent point sources. Consequently in Lloyd's mirror fringe width is not same for all parts of the source slit.

9.9 PHASE CHANGE ON REFLECTION: STOKE'S TREATMENT:

Suppose a light wave of amplitude 'a' is incident on an interface of two media as shown in Figure-9.7. Let r and t represent the fractions of the amplitude of the incident wave which are respectively reflected above and refracted below the surface. Hence the amplitudes of the reflected and refracted waves for an incident wave of amplitude a, will be 'ar' and 'at' respectively.

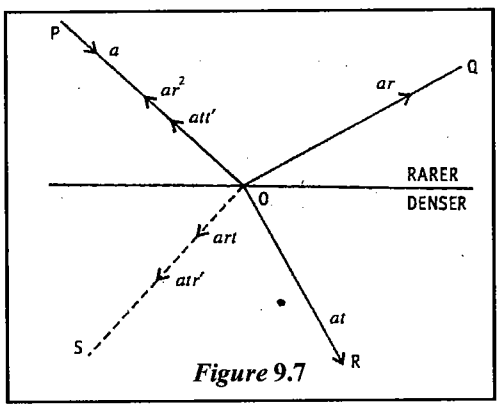


Figure 9.7

If there is no absorption of energy, the wave motion is a strictly reversible phenomenon. Now if the reflected wave along OQ having the amplitude atr' be reversed, we get one reflected wave of amplitude $ar \cdot r$ anlong OP and another refracted wave of amplitude $ar't$ along OS. Similarly if the wave 'at' along OR having the amplitude at be reversed, we get a reflected wave of amplitude atr' along OS and a refracted wave of amplitude att' along OP where r and t represent respectively the amplitude reflection and transmission coefficients when the wave is incident from lower medium. Since the resultant effect must consist of a wave along OP of amplitude a we have,

$$att' + ar^2 = a \qquad \ldots (32)$$

$$\Rightarrow r^2 + tt' = 1$$

The total amplitude along OS is zero, i.e.,

$$art + atr' = 0$$

$$\Rightarrow r' = -r \qquad \ldots (33)$$

The negative sign indicates that one of the rays has a positive displacement and the other has a negative displacement.

When reflection occurs from an interface beyond which the medium has a higher refractive index, the reflected wave undergoes a phase change of π. But there is no phase change occurs when reflection takes place from denser to rarer interface.

9.10 INTERFERENCE PHENOMENA IN THIN FILM, BASED ON DIVISION OF AMPLITUDE:

Beautiful colours are often exhibited when a beam of white light from an extended source is reflected from a thin film of oil floating on water or from a soap bubble. These phenomena can be explained on the basis of interference of light waves reflected from the upper and the lower surfaces of the film.

9.10.1 Interference in reflected light:

Let AB and CD be the bounding surfaces of a thin wedge-shaped film enclosing an angle α (Figure 9.8) Suppose a ray PQ of monochromatic light of wavelength λ is incident on the film. This ray will be partly refelcted along QO from the front surface and partly refracted along QR. The ray along QR after being partly reflected from the back surface CD, emerges along SO and meets the reflected ray QO at O.
The rays QO and SO are derived from the same incident ray PQ and hence are coherent. They combine to produce interference phenomena. Let us now calculate the phase difference between these reflected beams.

Draw $SN \perp^r QO$, $SN_1 \perp^r QR$ and $SLM \perp^r CD$. Produce QR and SL to meet at M. The paths of the two reflected beams (QO and SO) which are going to meet at O, will be equal from the dotted line SN upto O (for they are very close to each other). Hence the path difference of the two reflected interfering beams would be,

$$l = n(QN_1 + N_1R + RS) - QN \qquad \ldots (34)$$

where n = refractive index of the film.
From the Figure 9.8,
$RS = RM$ and $SL = LM = d$ = thickness of the film at S. By Snell's law we get,

$$n = \frac{\sin i}{\sin r} = \frac{QN/QS}{QN_1/QS} \qquad \Rightarrow QN = nQN_1 \qquad \ldots (35)$$

Thus Equation(35) now reduces to,

$$l = n(N_1R + RS) = n(N_1R + RM) = nN_1M$$

or, $$l = 2nd\cos(r - \alpha) \qquad \ldots (36)$$

In addition to this path difference, there is an extra phase difference of π equivalent to a path difference

$\pm\lambda/2$, caused by reflection at Q,

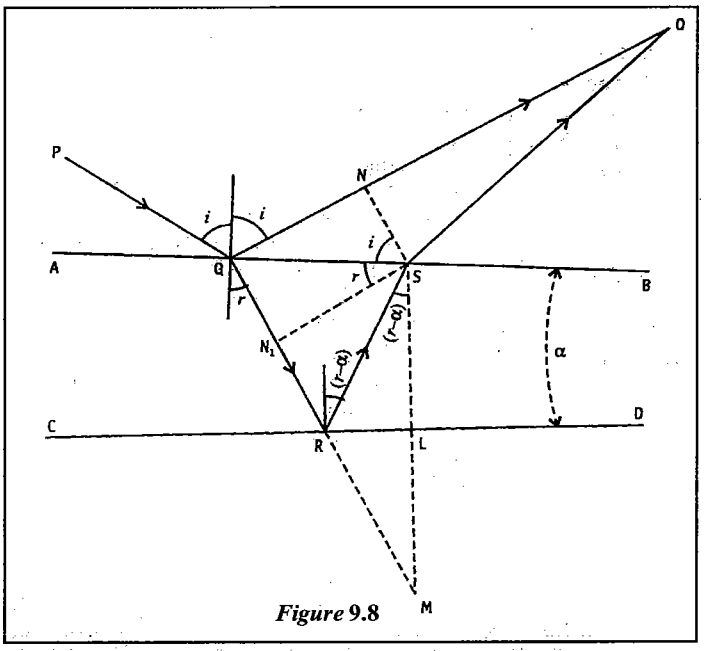


Figure 9.8

from the surface backed by denser medium. The reflection at R, from the surface backed by rarer medium, will not cause any change of phase.

Hence the total path difference between the two reflected rays is,

$$l = 2nd\cos(r-\alpha) \pm \frac{\lambda}{2} \quad \ldots (37)$$

- **Conditions for maxima and minima:**

For maxima of brightness at O,

$$2nd\cos(r-\alpha) \pm \frac{\lambda}{2} = n\lambda$$

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$$\Rightarrow \quad 2nd\cos(r-\alpha) = (2m+1)\frac{\lambda}{2} \text{ i.e. odd multiples of } \frac{\lambda}{2} \quad \ldots (38)$$

where m = 0, 1, 2, 3, ...

Similarly, for minima of brightness at O,

$$2nd\cos(r-\alpha) = m\lambda \quad \ldots (39)$$

- **Fringes with monochromatic light:**

(i) when a parallel beam of monochromatic light is incident on a film of varying thickness, n, λ and r are all constants. Hence different order fringes will be obtained for different thickness (d) of the film. At the thin edge of the film, where d is practically zero, the path difference (= 2nd cos r) of the two reflected rays inside the film is zero. **The only path difference between these two reflected rays is $\lambda/2$, caused by reflection from the surface AB of the film. Hence this thin edge would be dark not only for a light of one colour but for lights of all colours.**

As the thickness, d of the film increases, the order number m of the fringe also increases. Thus a fringe of a given order number will be the locus of the points where film thickness d is remaining same. If the film surfaces are perfectly plane, we get straight fringes parallel to the line of intersection of the surfaces of the film.

(ii) **When the film is extremely thin, so that d is practically zero for all parts of the film,** then the only path difference between the two reflected rays is $\lambda/2$ and **the film surface will appear perfectly dark even with white light.**

(iii) If the film surfaces are plane-parallel (i.e., $\alpha = 0$) and a parallel beam of monochromatic light is incident on its surface obliquely, then n, λ and d are all constants but r will very with the inclination of the parallel beam with the surface. Hence the film surface will be uniformly bright or dark according as the value of 'r' satisfies the conditions of brightness and darkness given in the relations (38) and (39) respectively. If the parallel beam is incident on such a film normally then n, d and λ are all constants and $\cos r = 1$. Hence the film surface will be uniformly bright or dark according as the thickness d satisfies the conditions of brightness or darkness.

- **Fringes with white light-colours of thin film:**

When a parallel beam of white light is incident on a thin wedge-shaped film, the values of n, λ and r will be different for lights of different colours. At the thinner edge of the film, thickness (d) is practically zero and this edge will be perfectly dark for light of all colours **following to the introduction of path difference of $\lambda/2$ by reflection from the surface AB of the film.** As $\lambda_v < \lambda_r$, the first order bright fringe of violet light will be formed at a smaller thickness of the film while the corresponding bright fringe of red light will be formed at the greater thickness. Thus we get differently coloured fringes at different thickness and these fringes are called fringes of equal chromatic order.

Beyond this point, we get another thickness where the condition of brightness will be simultaneously satisfied by two or more colours and here we get a coloured band due to the overlapping of these coloured bright fringes. When the thickness of the film is sufficient, the overlapping of differently coloured bright fringes goes to such an extent that we get uniform illumination. Thus we explain the colouration of thin films.

9.10.2 Interference in transmitted light:

A similar interference phenomenon is seen to occur with the rays transmitted through the film and emerging on the side CD of it (Figure 9.9). One part of the ray QR incident on the boundary CD, will emerge out of CD in the direction RO_1. Another part of it will also emerge out of CD along TO_1 after two successive internal reflections at R and S. These two internal reflections will cause no change of phase and hence the phase difference of these two emergent rays will be purely determined by their path difference.

In figure $NO_1 \approx TO_1$ and hence the path difference of the two transmitted interfering beams would be,

$$l = n(RN_1 + N_1S + ST) - RN \qquad \ldots (40)$$

Now, TN_1, TN and TLM are drawn perpendiculars on RS, RO_1 and AB respectively. RS and TL are produced to meet at M. **From the geometry of the Figure 9.9 we get, $ST = SM$ and $LT = LM = d =$ thickness of the film at L.**

By Snell's law we get, $n = \dfrac{\sin i_1}{\sin(r - \alpha)} = \dfrac{RN/RT}{RN_1/RT}$

or, $$RN = nRN_1 \qquad \ldots (41)$$

Thus Equation (40) reduces to,

$$l = n(N_1S + SM) = nN_1M = 2nd\cos(r - 2\alpha) \qquad \ldots (42)$$

Therefore, for maxima of brightness,

$$2nd\cos(r - 2\alpha) = 2m\frac{\lambda}{2} = m\lambda \qquad \ldots (43)$$

For minima of brightness,

$$2nd\cos(r - 2\alpha) = (2m + 1)\frac{\lambda}{2} \qquad \ldots (44)$$

For a film of very small wedge angle α or for a parallel film ($\alpha = 0$), then we note from the Equation (38), (39), (43) and (44) that the condition for maxima in the transmitted pattern corresponds to the condition for minima in the reflected pattern and vice-versa. Thus we may conclude that the fringes observed with the reflected and transmitted light are complementary to each other.

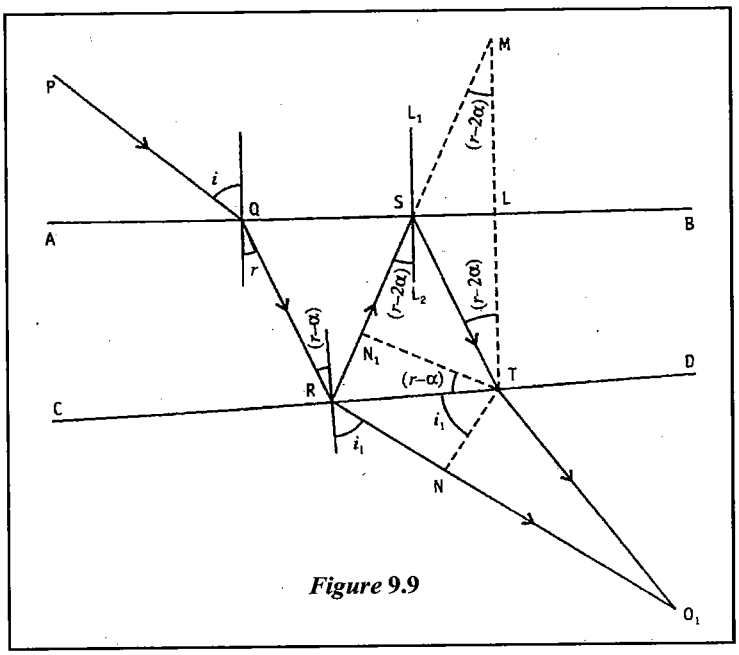


Figure 9.9

9.10.3 Necessity of using a broad source and thin film:

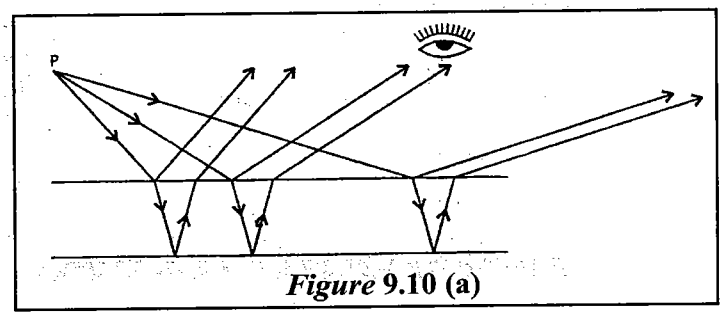


Figure 9.10 (a)

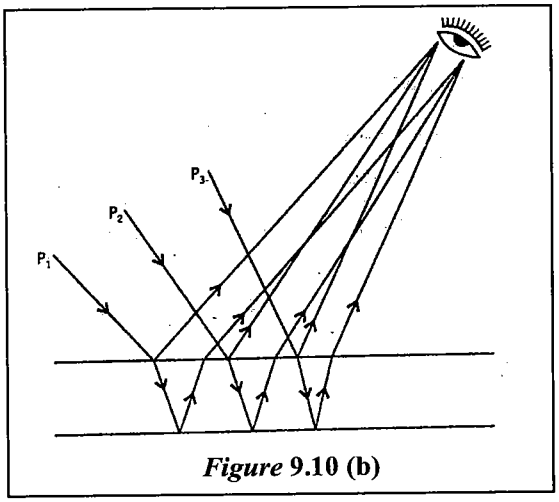


Figure 9.10 (b)

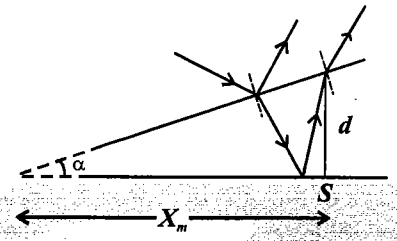
If rays from a point source P [Figure 9.10(a)] be made incident on the film then the reflected or transmitted rays that can enter the eye (whose aperture is small) are confined to a small range of directions. Hence field of view will be very small. On the other had, if a broad source of light is used, light from every point of the

source gives rise to a pair of coherent waves which can reach the eye after being reflected or transmitted from different points of the film. Thus the field of view become wide and fringes can be seen over the entire film [Figure 9.10(b).

For a thick film, the two adjacted reflected or transmitted rays will be wide apart and will not be able to enter the eye whose pupil is small. If the film be thin, then the two adjacent reflected or transmitted rays will be able to enter the eye where interference will occur by overlapping.

9.10.4 Fringe Width:

When a parallel beam of monochromatic light is incident normally on the film, the condition of brightness with reflected light is given by, $2nd = (2m + 1)\lambda / 2$ ... (45)



But the thickness d of the film at S is αx_m, where x_m is distance of the mth bright band from the apex of the film. Thus

$$2n\alpha x_m = (2m + 1)\lambda / 2. \quad \Rightarrow x_m = \frac{(2m + 1)\lambda}{4n\alpha} \quad \text{... (46)}$$

Similarly, $x_{m+1} = \dfrac{(2m + 2 + 1)\lambda}{4n\alpha}$ ... (47)

The distance (β) between two consecutive bright bands is given by,

$$\beta = x_{m+1} - x_m = \frac{\lambda}{2n\alpha} \quad \text{... (48)}$$

CAREER ENDEAVOUR

9. 10.5 Fringes of equal width and equal inclination:

The conditions for maxima and minima of brightness of the fringes formed by the light reflected from a thin wedge-shaped film are respectively given by,

$$2nd\cos(r - \alpha) = (2m + 1)\frac{\lambda}{2} \text{ (maxima)} \quad \text{... (49)}$$

$$2nd\cos(r - \alpha) = m\lambda \text{ (minima)} \quad \text{... (50)}$$

The above conditions suggest the existence of two distinct types of fringes. Suppose the incident rays are parallel and monochromatic and hence n, r and λ are all constants. Under this condition different order numbers (m) of the fringes will be controlled by the thickness d of the film. Hence **a fringe of a particular order number will lie on the locus of all the points of film having a constant thickness. These fringes are called fringes of equal width or thickness.**

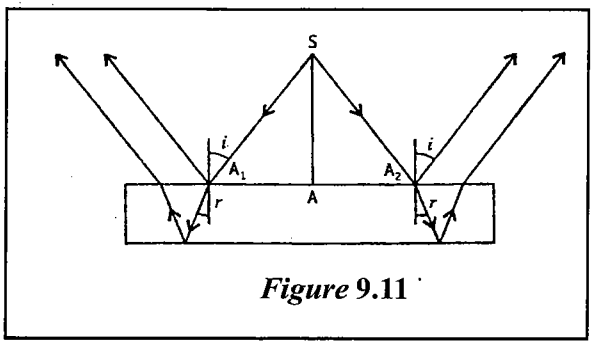


Figure 9.11

Another type of fringes will be obtained when monochromatic divergent rays from an extended source are incident on a film of uniform thickness. Here different order numbers (m) of the fringes will be controlled by the different values of r. If we draw a circle, with the cutting point A of the normal ray SA with the surface as centre, and AA_1 as radius, then all rays incident on the circumference of this circle will have the same angle of incidence i (Figure 9.11). Thus the fringe of a particular order will be a circle with the point A as centre. Fringe system will consist of concentric bright and dark rings. Fringe of a particular order is characterised by a particular angle of incidence and hence these fringes are called fringes of equal inclination. As the angle of incidence of the rays increases, the value of cos r decerases causing a decrease in the value of m. Thus the fringes of bigger radii will have smaller order number.

9.11 NON-REFLECTING FILM:

A part of the incident light is lost due to reflection at lens surfaces. Although the loss is small for a single surface, in a multi-lens system the loss is significant. Thus it is important to find ways of reducing or eliminating reflection is to use a thin layer of transparent material on the lens surface. This is known as anti-reflection coating.

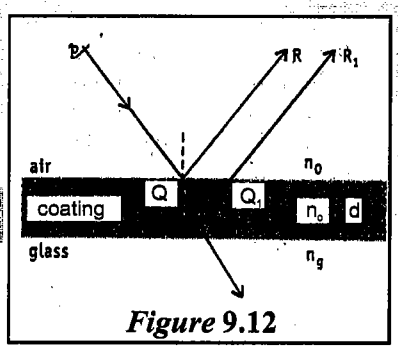


Figure 9.12

According to the electromagnetic theory of light the reflectivity r at the boundary between two media of refractive indices n_1 and n_2 and at normal incidence is given by

$$r = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 \qquad \ldots (51)$$

So for equality of the amplitudes of the waves along QR and Q_1R_1 we have

$$\left(\frac{n_c - n_0}{n_c + n_0}\right)^2 = \left(\frac{n_g - n_c}{n_g + n_c}\right)^2 \qquad \ldots (52)$$

Which, on simplification, shows that the refractive index of the coating must be

$$n_c = \sqrt{n_0\, n_g} \qquad \ldots (53)$$

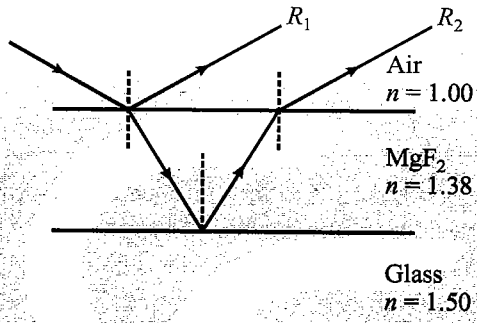
Here both the rays QR and Q_1R_1 suffer sudden phase change of π and hence for destructive interference the geometrical path difference between them must be equal to $\lambda/2$. Thus for normal incidence the minimum thickness 'd' of the coating is given by

$$2n_c d = \lambda/2 \quad \Rightarrow \quad d = \frac{\lambda}{4n_c} \qquad \ldots (54)$$

Most frequently used materials are **magnesium fluoride and cryolite**. The technique of reducing the reflectivity of a surface is known as blooming.

x. 17. A lens is coated with a thin film of transparent substance magnesium fluoride (MgF_2) with $n = 1.38$ to reduce the reflection from the glass surface ($n = 1.50$). How thick a coating is needed to produce a minimum reflection at the centre of the visible spectrum ($\lambda = 550$ nm) ?

ln. We assume that the light strikes the lens at near-normal incidence. We would like to find the thickness of the film that will bring about destructive interference between rays R_1 and R_2 as shown in the figure below.



Phase change of π is now associated with each ray and the condition of minimum intensity is now

$$2nd = \left(m + \frac{1}{2}\right)\lambda, \; m = 0, 1, 2, \ldots.$$

Thus, the thickness of the thin nest possible film is (for $m = 0$)

$$d = \frac{\lambda}{4n} = \frac{550}{4 \times 1.38} \text{ nm} = 99.64 \text{ nm}.$$

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. 18. A tanker leaks kerosene (n = 1.2) into the Persian Gulf, creating a large slick on top of the water ($n = 1.3$).

(a) If you are looking straight down from an airplane onto a region of the slick where its thickness is 460 nm, for which wavelength(s) of visible light is the reflection the greatest?

(b) If you are scuba-diving directly under this same region of the slick, for which wavelength(s) of visible light is the transmitted intensity the strongest.

ln. (a) For the reflected rays, the condition for maxima is

$$2nd = m\lambda, \text{ or } \lambda = \frac{2nd}{m} = \frac{2 \times 1.2 \times 460}{m} \text{nm}$$

The wavelength of visible light (for $m = 2$) is 552 nm.

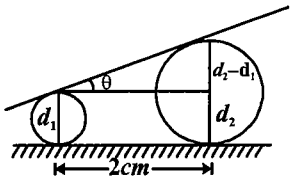
(b) For the transmitted rays, the condition for maxima is

$$2nd = \left(m + \frac{1}{2}\right)\lambda \text{ or } \lambda = \frac{2nd}{m + \frac{1}{2}} = \frac{2 \times 1.2 \times 460}{m + \frac{1}{2}} \text{nm}.$$

The wavelengths of visible light are 736 nm (when $m = 1$) and 441.6 nm (when $m = 2$).

Ex. 19. Two wires with diameters 0.01 cm and 0.03 cm are laid parallel to each other and 2 cm apart on a piece of plane glass. Another piece of plane glass is laid on top. Monochromatic light of wavelength 546 nm falls normally on the glass plates. Calculate the distance between consecutive bright fringes.

Soln.



Angle of the wedge,

$$\theta = \frac{d_2 - d_1}{\ell} = \frac{0.03 - 0.01}{2} = 0.01 \text{ rad.}$$

Fringe width,

$$\beta = \frac{\lambda}{2\theta} = \frac{546}{2 \times 0.01} \text{nm} = 27.3 \ \mu\text{m.}$$

9.12 NEWTON'S RINGS:

9.12.1 Introudction:

Newton's rings are a particular example of interference fringes formed by thin films. By placing a plano-convex lens on a plane glass plate a thin air film of progressively increasing thickness from the point of contact O can be formed as shown in Figure 9.13. If it is illuminated by a monochromatic light interference fringes in the form of concentric circular rings are found. These rings were first observed by Newton and are known as Newton's rings. These fringes are the loci of points of equal film thickness. These rings are localized in the air film.

9.12.2 Mathematical analysis:

The optical path difference between two successive reflected waves QS_1R_1 and NS_2R_2 as shown in Figure 9.13 will be $2nd \pm \lambda/2$ ... (55)

where d = thickness of the film at N. Here we assume the film to be optically rarer than the glass media, above and below and consequently we are to consider the abrupt phase change of π due to refection at G. This is equivalent to a path difference of $\pm\lambda/2$.

Case-I : For constructive interference

$$2nd \pm \frac{\lambda}{2} = m\frac{\lambda}{2}$$

$$\Rightarrow \quad 2nd = (2m+1)\lambda/2 \qquad \text{... (56)}$$

where m = 0, 1, 2, 3, ...

Case-II : For destructive interference,

$$2nd \pm \lambda/2 = (2m+1)\frac{\lambda}{2}$$

$$\Rightarrow \quad 2nd = m\lambda \qquad \text{... (57)}$$

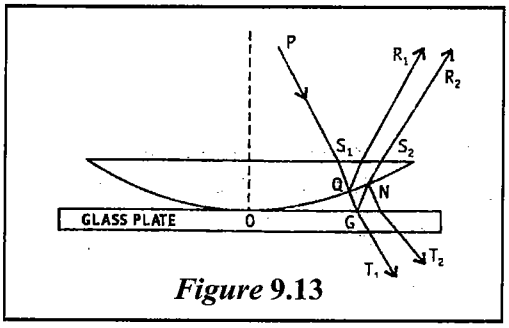


Figure 9.13

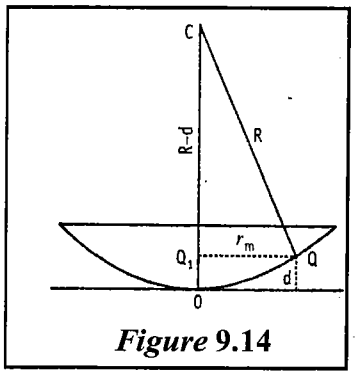


Figure 9.14

A fringe of a given order (m) will be along the loci of points of equal film thickness (d) and hence the fringes will be circular. The radii of the rings can be found out from Figure 9.14. If the point Q fulfils the condition of brightness (or darkness) then all points on the circumference of a circle of radius Q_1Q will be bright (or dark). Thus we shall get a bright (or dark) ring, of say m^{th} order, whose radius is $Q_1Q = r_m$. By geometry,

$$R^2 = r_m^2 + (R-d)^2 \qquad \ldots (58)$$

where R = radius of curvature of the convex surface.

Since R > > d, we can write $r_m^2 \approx 2Rd \Rightarrow 2d = \frac{r_m^2}{R}$ ... (59)

Using the condition (56) we can write for the m^{th} bright ring.

$$2nd = \frac{nr_m^2}{R} = (2m+1)\frac{\lambda}{2} \Rightarrow r_m^2 = \frac{(2m+1)\lambda R}{2n} \qquad \ldots (60)$$

Similarly for the m^{th} dark ring,

$$r_m^2 = \frac{2m.\lambda R}{2n} \qquad \ldots (61)$$

Thus we find that the radii of the bright rings are proportional to the square root of odd natural numbers and that of the dark rings proportional to square root of natural numbers. Corresponding diameters are given by

$$D_m^2 = \frac{2(2m+1)\lambda R}{n} \quad [\text{Bright ring}] \qquad \ldots (62)$$

$$D_m^2 = \frac{4m\lambda R}{n} \quad [\text{Dark ring}] \qquad \ldots (63)$$

The difference in diameters of the m^{th} and $(m+1)^{th}$ order dark rings is

$$D_{m+1} - D_m = \sqrt{\frac{4\lambda R}{n}}\left[\sqrt{m+1} - \sqrt{m}\right] \qquad \ldots (64)$$

Thus as the order number *'m'* increases this difference decreases meaning that the rings gradually become narrower as their radii increases. There will be more crowding of the rings as we move outward from the centre.

9.12.3 Fringe width:

If D_m and D_{m+1} are the diameters of two successive bright rings then we can write from Equation (63),

$$D_{m+1}^2 - D_m^2 = \frac{4\lambda R}{n} \Rightarrow D_{m+1} - D_m = \frac{4\lambda R}{n(D_{m+1} + D_m)}$$

Writing $D_{m+1} + D_m \approx 2D_m$, we get fringe width β as,

Fringe width, $$\beta = \frac{D_{m+1} - D_m}{2} = \frac{\lambda D}{n.D_m} \quad \ldots (65)$$

Thus fringe width β decreases as the diameter D_m of the ring increases.

9.12.4 Central fringe:

At the point of contact of the lens and the glass plate thickness of the film d = 0 Consequently, the condition (57) for destructive interference will be satisfied with m = 0. This indicates that the central fringe is dark and is situated at the point of contact of the lens and the glass plate. It appears as a dark spot.

9.12.5 Newton's rings with white light:

With a white light the central spot will be black and it will be surrounded by a few coloured rings and beyond this there will be general illumination due to overlapping of different coloured rings.

9.12.6 Determination of wavelength:

Newton's rings can be used for the measurement of the wavelength of monochromatic light. The diameters of m^{th} Newton's ring for an air film (n = 1) are given from Equations (62) and (63) as

$$D_m^2 = 2(2m + 1)\lambda R \text{ [Bright ring]} \quad \ldots (66)$$

$$D_m^2 = 4m\lambda R \text{ [Dark ring]} \quad \ldots (67)$$

Thus if D_m and D_{m+p} are respectively be the diameters of the m^{th} and $(m + p)^{th}$ rings (bright or dark) then the wavelength λ is given by

$$\lambda = \frac{D_{m+p}^2 - D_m^2}{4pR} \quad \ldots (68)$$

9.12.7 Determination of refractive index of a liquid:

Newton's rings can also be used to measure the refractive index (n) of a liquid. At first the diameters of m^{th} and $(m + p)^{th}$ bright or dark rings are measured with air film. Then the diameters of these rings are measured again by forming a liquid film between the lens and the glass plate. Introduction of the liquid decreases the diameters of the rings.

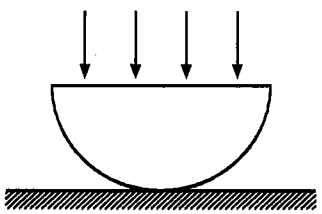
For air film, $$\left(D_{m+p}^2 - D_m^2\right)_{air} = 4p\lambda R$$

CAREER ENDEAVOUR

For liquid film, $$\left(D_{m+p}^2 - D_m^2\right)_{liquid} = \frac{4p\lambda R}{n}$$

Hence, $$n = \frac{\left(D_{m+p}^2 - D_m^2\right)_{air}}{\left(D_{m+p}^2 - D_m^2\right)_{liquid}} \quad \ldots (69)$$

Ex. 20. A thin slice is cut out of a glass cylinder along a plane parallel to its axis. The slice is placed on a flat plate as shown. The observed interference fringes from this combination shall be



(a) staright (b) circular (c) equally spaced
(d) having fringe spacing which increases as we go outwards

[IIT-JEE : 1999]

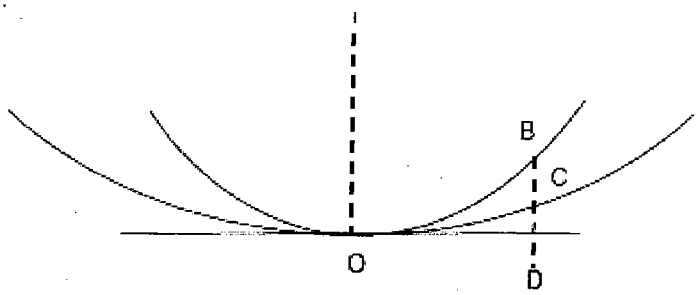
ln. **(a)**

Locus of equal path difference are the lines running parallel to the axis of the cylinder. Hence, straight fringes are obtained.

Note: Circular rings (also called Newton's rings) are observed in interference pattern when a plano-convex lens of large focal length is placed with its convex surface in contact with a plane glass plate because locus of equal path difference in this case is a circle.

. 21. The convex surface of radius 200 cm of a plano-convex lens rests on a concave spherical surface of radius 400 cm and Newton's rings are viewed with reflected light of wavelength 600 nm. Calculate the diameter of the 8th bright ring.

ln. The thickness of the air-film at B as shown in below figure.



$$t = BC = BD - CD = \frac{\rho^2}{2R_1} - \frac{\rho^2}{2R_2} = \frac{\rho}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

where $\rho = OD$, R_1 = Radius of curvature of OB and R_2 = Radius of curvature of OC. Thus, the condition of the mth bright fringe is

$$\rho_m^2 = \frac{\left(m - \frac{1}{2}\right)\frac{\lambda}{n}}{\frac{1}{R_1} - \frac{1}{R_2}}$$

In the present problem, $m = 8$, $\lambda = 600$ nm, $n = 1$, $R_1 = 200$ cm, $R_2 = 400$ cm. We find $2\rho_8 = 0.85$ cm.

9.12.8 Michelson's interferometer:

An instrument that is designed primarily to measure lengths or changes in length with degree of accuracy by means of interference fringes is known as an interferometer. One of the widely used interferometers is the one devised and built by A.A. Michelson.

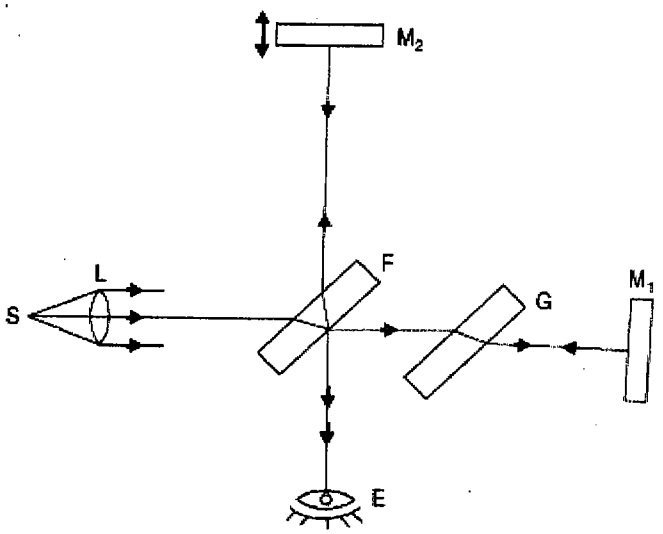


Figure **9.15**

The light from the source S falls on a beam splitter F (Fig. 9.15). F and G are two plane-parallel optical flats, which are usually cut from a single parallel plate so that they are identical. The surface of F nearest to G may be half silvered in order that the light falling upon it be 50% reflected and 50% transmitted, although this is not necessary for the operation of the interferometer. The mirror M_2 is mounted on a carriage, so that it can be moved along a precision-mechanical track. The motion of M_2 is controlled by a very fine micrometer screw. The mirror M_1 is held by springs against adjusting screws so that M_1 may be made exactly perpendicular to M_2, Light from the source S is rendered parallel by the lens L. After entering F the light is divided into two parts: one part proceeds to the mirror M_1 which returns the light to the eye after reflection at F; the other part, reflecting inside F, proceeds to M_2, which reflects it back through F to E. From a single source, by division of amplitude, two beams are produced and they are recombined for the formation of interference fringes.

The beam going towards the mirror M_2 and reflected back, has to pass twice through the glass plate F. Therefore to compensate for the path the plate G is used between the mirror M_1 and F. The light beam going towards the mirror M_1 and reflected back towards F also passes twice through the compensating plate G. Thus the paths of the two rays in glass are the same. Suppose the mirrors M_1 and M_2 are at distances d_1 and d_2 respectively from the glass plate F. In measuring d_1 and d_2 we do not consider the paths of the rays in glass. The path difference for the two waves when they recombine is $2d_2 - 2d_1$ and anything that changes this path difference will cause a change in the relative phase of the two waves as they enter the eye. If the mirror M_2 is moved by a distance $\frac{1}{2}\lambda$, the path difference changes by » and the observer will see the fringe pattern shift by one fringe. If the mirrors M_1 and M_2 are exactly perpendicular to each other, the system is essentially equivalent to the interference from an air film whose thickness is equal to $d_2 - d_1$. Suppose when the mirror M_2 is moved through a distance d, the number of fringes that cross the field of view is m. Then we have $d = m\frac{\lambda}{2}$ **or** $2d = m\lambda$.

Ex. 22. An airtight chamber 5 cm long with glass windows is placed in one arm of a Michelson's interferometer. Light of wavelength $\lambda = 480$ nm is used. The air is slowly evacuated from the chamber using a vacuum pump. While the air is being removed 61 fringes are observed to pass through the field of view. Find the index of refraction of air at atmospheric pressure.

Soln. In Michelson interferometer, light travels twice through the chamber. Thus, we have the formula,

$$2t = \frac{x\lambda}{(n-1)\beta} = \frac{61\beta\lambda}{(n-1)\beta} \text{ or } n-1 = \frac{61\lambda}{2t} = \frac{61\times 480\times 10^{-9}}{2\times 5\times 10^{-2}} = 0.00029$$

giving $n = 1.00029$.

Ex. 23. After obtaining the fringes in Michelson's interferometer with white light, the white light is replaced by sodium light. When one mirror of the interferometer is now moved through a distance of 0.15 mm the fringes are found to disappear. If the mean wavelength for the two components of the D lines of sodium light is 5893 Å, find the difference between their wavelengths.

Soln. We have $2d = m_1\lambda_1 = \left(m_1 + \frac{1}{2}\right)\lambda_2$, which gives $m_1 = \frac{\lambda_2}{2(\lambda_1 - \lambda_2)}$ and $\Delta\lambda = \lambda_1 - \lambda_2 = \frac{\lambda_1\lambda_2}{4d} = \frac{\lambda^2}{4d}$

$$= \frac{5893\times 5893\times 10^{-8}}{4\times 0.15\times 10^{-1}} \text{ Å} = 5.79 \text{ Å}.$$

Ex. 24. When a liquid is introduced between the lens and the plate in the Newton Ring apparatus, the diameter of the 15^{th} ring changes from 2cm to 1.8cm. The refractive index of the liquid is **[DU 2015]**

(a) 1.05 (b) 1.11 (c) 1.23 (d) 1.33

ln. $D \propto \frac{1}{\sqrt{\mu}}$ $\mu \rightarrow$ refractive index

$$\Rightarrow \quad \frac{D_2}{D_1} = \sqrt{\frac{\mu_1}{\mu_2}} \Rightarrow \frac{1.8\ cm}{2cm} = \sqrt{\frac{1}{\mu}}$$

$$\Rightarrow \quad \mu = \frac{2^2}{1.8^2} = \left(\frac{10}{9}\right)^2 \approx 1.23$$

. 25. The spherial surface of a plano-convex lens of radius of curvature R = 1m is gently placed on a flat plate. The space between them is filled with a transparent liquid of refractive indices of the lens and the flat plate are 1.5 and 1.6

[IIT-JAM : 2006]

n. (a) Optical path difference = $2\mu t$, Where μ is refractive index of liquid.
For destructive interference, and t is thickness of the film at A.

$$2\mu t = \left(n + \frac{1}{2}\right)\lambda \qquad \ldots (1)$$

and $2t = \frac{\rho_n^2}{R}$ ... (2) Where ρ_n is radius of n^{th} dark ring.

Substituting (2) into (1)

$$\frac{\mu\rho_n^2}{R} = \left(n + \frac{1}{2}\right)\lambda \quad \Rightarrow \rho_n = \sqrt{\left(n + \frac{1}{2}\right)\frac{\lambda R}{\mu}}$$

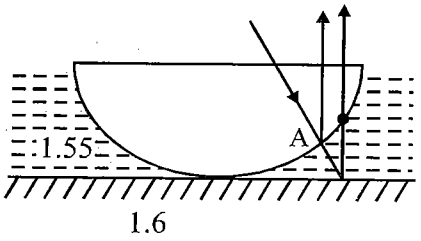
ρ_{16} = radius of 16^{th} dark ring $= \sqrt{5} \times 10^{-3}$ m.

16^{th} dark ring will correspond to n = 15.

$$\sqrt{5} \times 10^{-3} = \sqrt{\frac{31}{2}} \times \frac{\lambda \times 1}{\mu}$$

$$\lambda = \frac{5 \times 10^{-6} \times 1.55}{15.5} = 5 \times 10^{-7} \text{ met}$$

$$= 0.5\ \mu m$$



(b) When transparent liquid is removed, only ray 2 suffers a phase change of π, and path difference $\frac{\lambda}{2}$.

$$\text{Path difference} = \left(2t + \frac{\lambda}{2}\right)$$

For dark ring $2t + \frac{\lambda}{2} = \left(n + \frac{1}{2}\right)\lambda \quad \Rightarrow 2t = n\lambda \qquad \ldots (3)$

From (3), $\frac{\rho_n^2}{R} = n\lambda \quad \Rightarrow \rho_n = \sqrt{nR\lambda}$

For 21^{th} dark ring, n = 20 $\quad \Rightarrow \rho_{20} = \sqrt{20 \times 5 \times 10^{-11} \times 1} = \sqrt{10}$ mm

SOLVED EXAMPLES

1. In a Lloyd's mirror experiment calculate the ratio of ratio of the intensities at the interference maxima and minima if the mirror reflects only 75% of the light incident upon it.

Soln. Reflectivity R = 75 / 100 = 3 /4. Amplitude reflection coefficient $r = \sqrt{R} = \sqrt{3}/2$. So a_1 be the amplitude of the direct beam then the amplitude of the reflected beam would be $a_2 = a_1 r = a_1\sqrt{3}/2$

$$\Rightarrow \quad \frac{I_{max}}{I_{min}} = \frac{(a_1 + a_2)^2}{(a_{1-a_2})^2} = \frac{(1+r)^2}{(1-r)^2} = \left(\frac{1+\sqrt{3}/2}{1-\sqrt{3}/2}\right)^2 = 194$$

2. The acute angle of a biprism of refractive index 1.5 is 2°. A slit illuminated by a monochromatic light is placed 10 cm from the biprism. If the distance between the two dark fringes observed at a distance of 1 cm from the biprim is 0.18 mm, find the wavelength of light used.

Soln. We know that fringe width $\beta = \frac{\lambda D}{d}$

where the distance d between virtual sources is given by $d = 2a(n-1)\alpha$

Here a = 10 cm = 0.1 m, n = 1.5, $\alpha = 2° = 2\lambda/180$ radian.

$$\Rightarrow \quad d = 2\times 0.1(1.5-1)\times 2\pi/180 = 0.349\times 10^{-2}\text{ m}$$

Also, $\beta = 0.18\times 10^{-3}$ m, D = 1m + 0.1m = 1.1m.

$$\Rightarrow \quad \lambda = \frac{0.18\times 10^{-3}\times 0.349\times 10^{-2}}{1.1}\text{ m} = 571.2\text{ nm}$$

3. In Newton's ring experiment, the diameter of *m*th dark ring is 8 mm and the diameter of (m + 5)th dark ring is 12 mm. If the radius of curvature of the lower surface of the lens is 10 m, find the wavelength of light used.

Soln. We know that $\lambda = \frac{D_{m+p}^2 - D_m^2}{4pR}$

Here p = 5, R = 10 m, $D_m = 8\times 10^{-3}$ m

$$D_{m+p} = 12\times 10^{-3}\text{m}$$

$$\Rightarrow \quad \lambda = \frac{(12\times 10^{-3})^2 - (8\times 10^{-3})^2}{4\times 5\times 10}\text{ m} = 400\text{ nm.}$$

4. Newton's ring are formed in reflected light of wavelength 600 nm with a combination of plane plate of glass and a plano-convex lens of 1m radius of curvature. On introducing a liquid between the lens and the plate it is found that the diameter of the 7th dark ring decreases by 0.54 mm. Find the refractive index of the liquid.

Soln. The diameter D_m of *m*th dark ring is given by,

$$D_m^2 = 4m\lambda R \quad \text{when the film is air.}$$

$$\Rightarrow \quad D_7 = \sqrt{4\times 7\times 600\times 10^{-9}\times 1}\text{ m} = 4.1\text{ mm}$$

When there is a liquid of refractive index n within the plate and lens, the diameter becomes (4.1 – 0.54) mm = 3.56 mm.

Now $\left(D_m^2\right)_{liquid} = \dfrac{4m\lambda R}{n}$

Therefore, refractive index

$$n = \frac{\left(D_m^2\right)_{air}}{\left(D_m^2\right)_{liquid}} = \frac{(4.1)^2}{(3.56)^2} = 1.326$$

5. In Young's double slit experiment the angular position of a point above the central maximum whose intensity is one-fourth of maximum intensity is

(a) $\sin^{-1}\left(\dfrac{\lambda}{d}\right)$ (b) $\sin^{-1}\left(\dfrac{\lambda}{2d}\right)$ (c) $\sin^{-1}\left(\dfrac{\lambda}{3d}\right)$ (d) $\sin^{-1}\left(\dfrac{\lambda}{4d}\right)$

Soln. $I = I_{\max}\cos^2\left(\dfrac{\delta}{2}\right)$ where $\delta = \dfrac{2\pi}{\lambda}d\sin\theta$

$\dfrac{1}{4} = \cos^2\dfrac{\delta}{2}$ or $\dfrac{\delta}{2} = \dfrac{\pi}{3}$ or $\delta = \dfrac{2\pi}{3}$.

Thus, $\dfrac{2\pi}{3} = \dfrac{2\pi}{\lambda}d\sin\theta$ **or** $\sin\theta = \dfrac{\lambda}{3d}$ **or** $\theta = \sin^{-1}\left(\dfrac{\lambda}{3d}\right)$.

Hence, correct option is (c).

. In Newton's experiment with a light containing two wavelengths $\lambda_1 = 589\,\text{nm}$ and $\lambda_2 = 589.6\,\text{nm}$. find the distance (from the point of contact of plano-convex lens with glass plate) at which the rings disappear. Take the radius of curvature of the curved surface as 1 m.

oln. The fringes will disappear if mth order bright ring of λ_1 overlaps with mth order dark ring of λ_2.

We know,

$$D_m^2(\text{bright}) = \frac{2(2m+1)\lambda, R}{n} \quad \ldots (i)$$

$$D_m^2(\text{dark}) = \frac{4m\lambda R}{n} \quad \ldots (ii)$$

$$D_m^2(\text{bright}) = D_m^2(\text{dark})$$

$$\Rightarrow \quad \frac{2(2m+1)\lambda_1 R}{n} = \frac{4m\lambda_2 R}{n}$$

$$\Rightarrow \quad (2m+1)\lambda_1 = 2m\lambda_2$$

$$\Rightarrow \quad \lambda_1 = 2m(\lambda_2 - \lambda_1)$$

$$\Rightarrow \quad m = \frac{\lambda_1}{2(\lambda_2 - \lambda_1)}$$

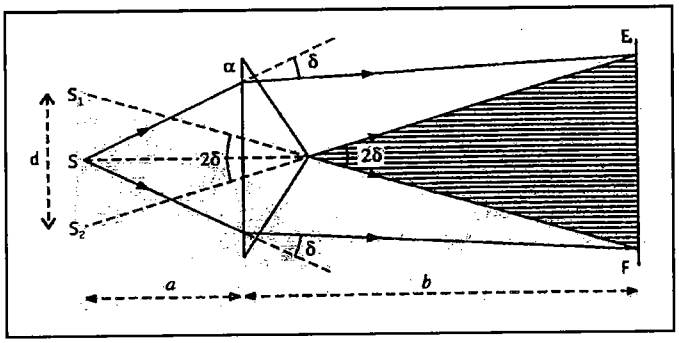
$$\Rightarrow \quad m = \frac{\lambda_1}{2(\lambda_2 - \lambda_1)} = \frac{589}{2 \times 0.6} = 490$$

Therefore, the distance at which the rings will disappear is

$$r_{490} = \sqrt{490 \times 589.6 \times 10^{-9} \times 1}\ \text{m} \approx 1.7\,\text{m}$$

7. In a Fresnel's biprism experiment the base angle of the prism is 3° and the refractive index of glass is 1.5. Interference fringes are formed with a source of wavelength 600 nm located 10 cm from the biprism. If the screen is placed at a distance of 1 m from the source find the maximum number of fringes that can be observed.

Soln.



In Figure above, deviation $\delta = (n-1)\alpha$

Distance between virtual sources is $d = 2\delta.a = 2a(n-1)\alpha$

Fringe width $\beta = \dfrac{\lambda(a+b)}{2a(n-1)\alpha} = \dfrac{\lambda D}{d}$ $\quad [\because D = a+b]$

Overlapping region $EF = b.2\delta = 2b(n-1)\alpha$

$\therefore$ Maximum number of fringes observable is given by $\dfrac{EF}{\beta}$.

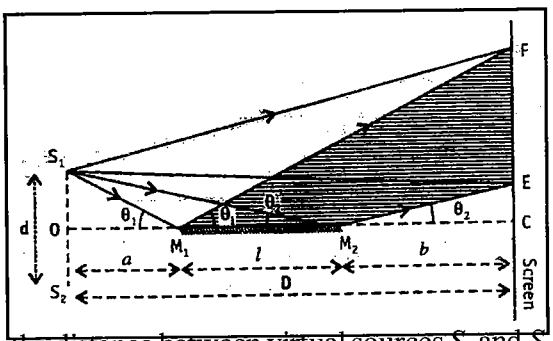
Here a = 10 cm, b = 90 cm, a + b = 1m, $\lambda = 600 \times 10^{-9}$ m, $n = 1.5$, $\alpha = 3^\circ = \dfrac{3 \times \pi}{180}$ radian.

$$\Rightarrow \quad \beta = \frac{600 \times 10^{-9} \times 1 \times 180}{2 \times 0.1 \times 0.5 \times 3\lambda}\text{m} = 11.46 \times 10^{-5}\text{m}$$

Hence maximum number of fringes observable would be $\dfrac{EF}{\beta} = \dfrac{2b(n-1)\alpha}{\beta} = \dfrac{2 \times 0.9 \times 0.5 \times 3\pi}{11.46 \times 10^{-5} \times 180} = 411$

8. A Lloyd's mirror of length 5 cm is illuminated with a light of wavelength 589 nm from a narrow slit 0.1 cm from its plane and 5 cm measured in that plane, from its near edge. Find the fringe width on a screen placed 1.2 m from the slit. Also, determine the maximum number of observable fringes.

Soln.



In Figure above the distance between virtual sources S_1 and S_2 is

$d = 2 \times 0.1$ cm = 0.2 cm = 0.2×10^{-2} m and D = 1.2 m

$\Rightarrow$ Fringe width $\beta = \frac{\lambda D}{d} = \frac{589 \times 10^{-9} \times 1.2}{0.2 \times 10^{-2}}$ m $= 0.3534$ mm

In Figure above, EF is the region of overlap on the screen. Now

$$CE = b\tan\theta_2 = (D - a - l)\tan\theta_2 = (D - a - l).\frac{d/2}{(a + l)}$$

$$CF = (b + l)\tan\theta_1 = (D - a)\tan\theta_1 = (D - a).\frac{d/2}{a}$$

Here a = 0.05 m, l = length of mirror = 0.05 m, D = 1.2 m

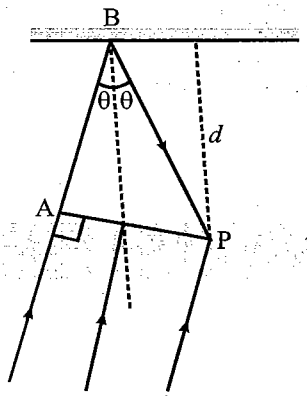
$\Rightarrow$ $$EF = CF - CE = (D - a)\frac{d}{2a} - (D - a - l).\frac{d}{2(a + l)}$$

$$= (1.2 - 0.05)\frac{0.002}{2 \times 0.05} - (1.2 - 0.05 - 0.05)\frac{0.002}{2 \times 0.1} = 0.012 \text{ m} = 1.2 \text{ cm}$$

Therefore, maximum number of fringes observable is $\frac{EF}{\beta} = \frac{12mm}{0.3534mm} = 34$

A plane wavefront of light is incident on a plane mirror as shown in below figure. Show that the intensity is maximum at P when

$$\cos\theta = \frac{\lambda}{4d}, \frac{3\lambda}{4d}, \frac{5d}{4d}, \ldots.$$



oln. The path difference between the disturbance reaching at point P directly and after reflection is

$$AB + BP + \frac{\lambda}{2} = BP\cos 2\theta + \frac{d}{\sin(90° - \theta)} + \frac{\lambda}{2} = \frac{d}{\cos\theta}(\cos 2\theta + 1) + \frac{\lambda}{2} = 2d\cos\theta + \frac{\lambda}{2}$$

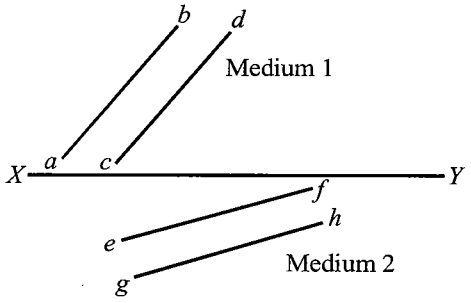
Here $\frac{\lambda}{2}$ is due to reflection from the denser medium.

For maximum intensity at P, we have

$$2d\cos\theta + \frac{\lambda}{2} = n\lambda, n = 1, 2, 3, \ldots$$

or $$\cos\theta = \frac{(2n - 1)\lambda}{4d}, n = 1, 2, 3, \ldots$$

10. The below figure shows a surface XY separating two transparent media, medium 1 and medium 2. The lines ab and cd represent wavefronts of a light wave travelling in medium 1 and incident on XY. The lines ef and gh represent wave fronts of the light wave in medium-2 after refraction.



(a) Light travels as a
(A) parallel beam in each medium
(B) convergent beam in each medium
(C) divergent beam in each medium
(D) divergent beam in one medium and convergent beam in the other medium.

(b) The phases of the light wave at c, d, e, f are ϕ_c, ϕ_d, ϕ_e and ϕ_f respectively. It is given that $\phi_c \neq \phi_f$. Then

(A) ϕ_c cannot be equal to ϕ_e (B) ϕ_d cannot be equal to ϕ_e

(C) $(\phi_d - \phi_f)$ is equal to $(\phi_c - \phi_e)$ (D) $(\phi_d - \phi_c)$ is not equal to $(\phi_f - \phi_e)$

(c) Speed of light is
(A) the same in medium 1 and medium 2
(B) larger in medium 1 than in medium 2
(C) larger in medium 2 than in medium 1
(D) different at b and d.

Soln. (a) The ray of light is perpendicular to the wavefront.
Correct choice (A)

(b) The particles are in same phase on a wavefront.

Thus $\phi_c = \phi_d$ and $\phi_e = \phi_f$

$$\phi_d - \phi_f = \phi_c - \phi_e$$

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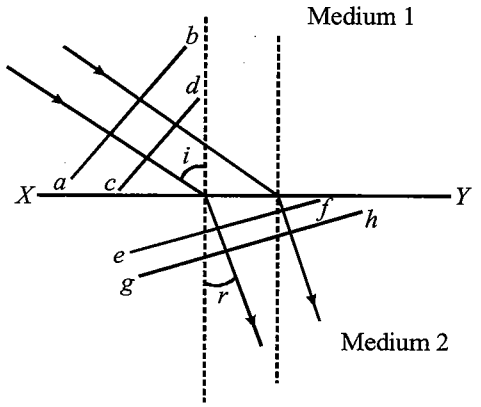
Correct choice (C)

(c) The ray of light goes towards normal after refraction (see below figure). Thus the medium 2 is denser than medium 1. The velocity of light decreases in the denser medium.



Correct choice (B).

1. In a double slit experiment instead of taking slits of equal widths, one slit is made twice as wide as the other, then in the interference pattern
(a) the intensities of both the maxima and the minima increases
(b) the intensities of the maxima decreases and the minima has zero intensity
(c) the intensity of maxima decreases and that of minima increases
(d) the intensity of maxima decreases and the minima has zero intensity

[IIT-JEE : 2000]

oln. (a)

In interference we know that

$I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$ and $I_{min} = \left(\sqrt{I_1} \sim \sqrt{I_2}\right)^2$

Under normal conditions (when the widths of both the slits are equal)

$I_1 \approx I_2 = I$ (say)

$\therefore \; I_{max} = 4I$ and $I_{min} = 0$

When the width of one of the slits is increased. Intensity due to that slit would increase, while that of the other will remain same. So, let :

$I_1 = I$ and $I_2 = \eta I \qquad (\eta > 1)$

Then, $I_{max} = I\left(1 + \sqrt{\eta}\right)^2 > 4I$

and $I_{min} = I\left(\sqrt{\eta} - 1\right)^2 > 0$

$\therefore$ Intensity of both maxima and minima is increased.

:. Find the thickness and refractive index of an anti-reflection coating necessary for a glass lens ($n_g = 1.5$) in air at vacuum wavelength 590 nm.

ln. The thickness of the anti-reflection coating is chosen so that the wavelength in the dielectric material is one quarter the wavelength of the incoming wave. For a quarter wavelength antireflection coating of a transparent material with a refractive index n_c and light incident on the coating with a free-space wavelength λ_0, the thickness t which causes minimum reflection is calculated by $\frac{\lambda_c}{4} = \frac{\lambda_0}{4n_c}$

Reflection is further minimized if the refractive index of the anti-reflection coating is the geometric mean of that of the materials on either side; that is, glass or air. Refractive index of the coating material is given by

$n_c = \sqrt{n_0 n_g} = \sqrt{1 \times 1.5} = 1.225$

Thickness required is equal to

$\Rightarrow \; t = \frac{\lambda_0}{4n_c} = \frac{590}{4 \times 1.225}$ nm = 120.4 nm.

Chapter 10

Diffraction of Light

10.1 FRAUNHOFER DIFFRACTION IN A SINGLE SLIT:

Let a plane wavefront AB of monochormatic light of wavelength λ propagating normal to a slit S_1S_2 be incident on it. According to Huygens-Fresnel principle each point of the wavefront in the plane of the slit may be regarded as the source of secondary spherical wavelets. The secondary wavelets travelling normal to the slit are brought to focus by a convex lens L on screen at C. The wavelets travelling at an angle θ with the normal are brought to focus at Q. Let us calculate the intensity of light at Q.

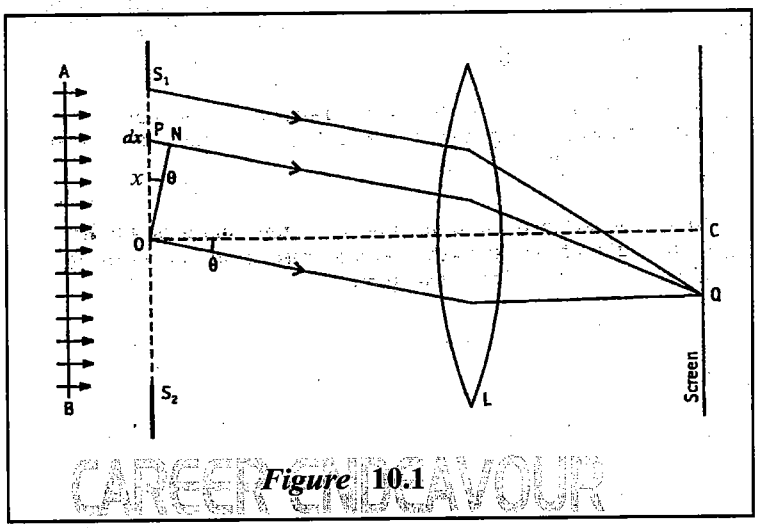


Figure 10.1

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10.1.1 Intensity of the diffraction spots:

Let the complex light disturbance at any instant due to secondary waves from the origin O (the mid point of the slit) be represented by $Ae^{i\omega t}$ where A is the amplitude and ω is the circular frequency of the wave. The phase difference between the waves at Q coming from O and from a point P at a distance x from O is given by

$$\frac{2\pi}{\lambda} \times PN = \frac{2\pi}{\lambda} x \sin\theta = lx \qquad \ldots (1)$$

where, $$l = \frac{2\pi}{\lambda} \sin\theta \qquad \ldots (2)$$

Hence the disturbance at Q due to secondary waves from P will be proportional to $e^{i(\omega t - lx)}$. The disturbance at Q to the diffracting element dx can be written as

$$dy = C.A.dx.e^{i(\omega t - lx)} \qquad \ldots (3)$$

where we assume the amplitude to be proportional to the width dx; C is a proportionality constant.
If a is the width of the slit then the resultant complex disturbance at Q due to the waves coming from all the diffracting elements will be given by

$$y = \int_{-\frac{a}{2}}^{+\frac{a}{2}} CAe^{i(\omega t - lx)}dx = CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{-\frac{a}{2}}^{+\frac{a}{2}} = CAa.\frac{\sin\frac{la}{2}}{\frac{la}{2}}.e^{i\omega t} \quad \ldots (4)$$

$$\left[\because \sin\theta = \frac{e^{i\theta} - e^{i\theta}}{2i}\right]$$

The resultant intensity I at Q is obtained by multiplying y by its complex conjugate y^*. Thus

$$I = y.y^* = I_0\frac{\sin^2\alpha}{\alpha^2} \quad \ldots (5)$$

where $I_0 = (CAa)^2$ and $\alpha = \frac{la}{2} = \frac{\pi}{\lambda}.a\sin\theta$ ... (6)

10.1.2 Condition for maxima and minima:

The maxima or minima are given by $\frac{dI}{d\alpha} = 0$

$$\Rightarrow \quad \frac{2\sin\alpha\cos\alpha}{\alpha^2} - \frac{2\sin^2\alpha}{\alpha^3} = 0$$

$$\Rightarrow \quad \sin\alpha(\alpha\cos\alpha - \sin\alpha) = 0$$

Hence, either $\sin\alpha = 0$ ... (7)

or $\alpha = \tan\alpha$ ... (8)

The value of $\frac{d^2I}{d\alpha^2}$ is found to be positive for $\sin\alpha = 0$ and negative for $\alpha = \tan\alpha$. Hence the condition (7) will give minima and condition (8)will give maxima. Thus for minima,

$$\sin\alpha = 0$$

$$\Rightarrow \quad \alpha = m\pi,\ m = \pm1, \pm2, \pm3.....$$

$$\Rightarrow \quad a\sin\theta = m\lambda \quad \ldots (9)$$

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Here $m = 0$ is excluded because for this $\alpha = 0$ and since $\lim_{\alpha\to0}\frac{\sin\alpha}{\alpha} = 1$, it actually gives a maximum.

The positions of maxima can be obtained by solving Equation (8) It is transcendental equation and can be solved graphically by plotting the curves $y = \alpha$ and $y = \tan\alpha$ and finding the point of intersections as shown in Figure (10.2). An inspection of the Figure (10.2) shows that one value is $\alpha = 0$ and other values of α which will give maxima are less than but gradually approaching towards $\pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ ... etc.

$\alpha = 0$ gives the position of principal maximum and the other values of α give secondary maxima.

The intensity of principal maximum is

$$I = \lim_{\alpha\to0} I_0\frac{\sin^2\alpha}{\alpha^2} = I_0 \quad \ldots (10)$$

The intensity of first secondary maximum is

$$I_1 \approx I_0\frac{\sin^2\frac{3\pi}{2}}{\left(\frac{3\pi}{2}\right)^2} = \frac{4}{9\pi^2}I_0 \quad \ldots (11)$$

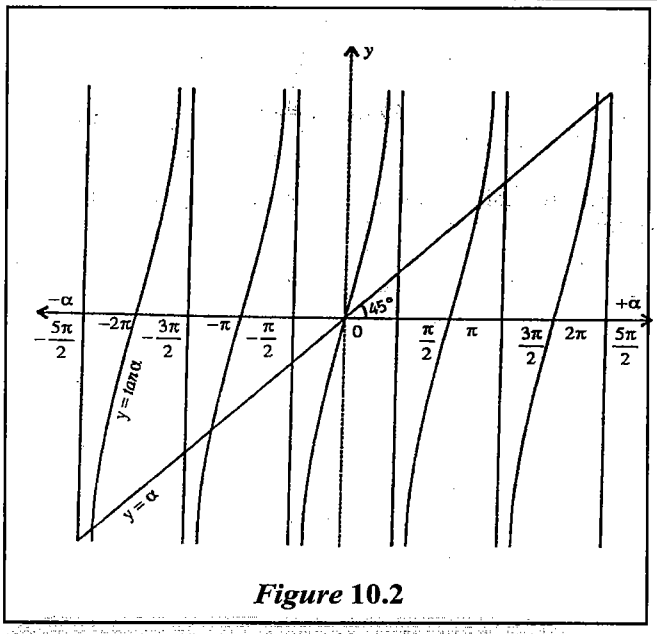


***Figure* 10.2**

The intensity of second secondary maximum is

$$I_2 \approx I_0 \frac{\sin^2 \frac{5\pi}{2}}{\left(\frac{5\pi}{2}\right)^2} = \frac{4}{25\pi^2} I_0 \qquad \ldots (12)$$

Thus the secondary maxima are very feeble compared with the principal maximum and their intensities diminish very rapidly with the increase in order number.
The intensity distribution in the diffraction pattern due to a single slit is shown in Figure (10.3).

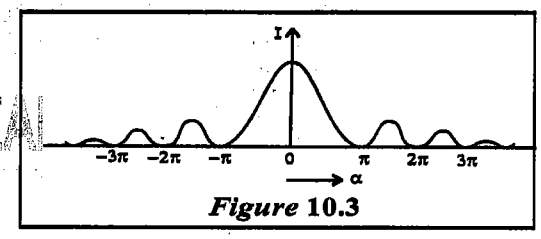


***Figure* 10.3**

The angle of the diffraction θ_1 for the first minima on either side of central maximum is given by Equation(9) as

$$a \sin\theta_1 = \lambda; \quad \Rightarrow \theta_1 \approx \sin\theta_1 = \frac{\lambda}{a}$$

Therefore, the angular width of principal maximum $2\theta_1 = 2\lambda / a$, is inversely proportional to the width (a) of the slit.

Ex. 1. In the Fraunhofer diffraction of light of wavelength λ at a slit of width a, the angular positions of different diffraction maxima (other than the central maximum) are given by

(a) $\sin\theta = \pm\frac{m\lambda}{a}$; $m = 0, 1, 2, 3, \ldots.$ (b) $\sin\theta \cong \pm\left(m + \frac{1}{2}\right)\frac{\lambda}{a}$; $m = 0, 1, 2, 3, \ldots.$

(c) $\sin\theta = \pm\frac{m\lambda}{a}$; $m = 1, 2, 3, \ldots.$ (d) $\sin\theta \cong \pm\left(m + \frac{1}{2}\right)\frac{\lambda}{a}$; $m = 1, 2, 3, \ldots.$

[IIT-JAM GP : 2005]

ln. (b)

. 2. Monochromatic light of wavelength 633nm falls on a slit. If the angular separation between the first minimum of intensity on either side of the central maximum is 19.5 degrees, the slit width is approximately,

(a) 3.7 mm (b) 37 μm (c) 3.7 μm (d) 37 nm

[IISc : 2006]

ln. (c)

Angular separation, $\theta = \frac{1}{2} \times 19.5 = 9.75°$

The position of first minima after central maxima is given by

$\Rightarrow a = \frac{\lambda}{\sin\theta} = 3.7\ \mu m$

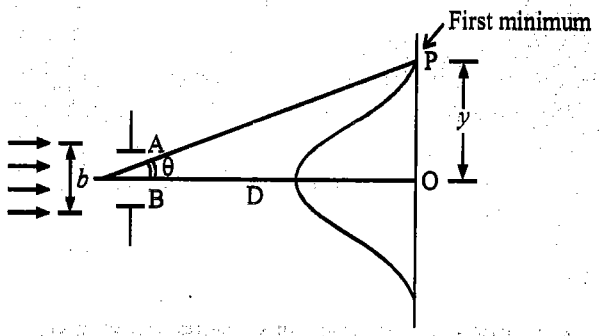
. 3. A parallel monochromatic beam of light is incident normally on a narrow slit. A diffraction pattern is formed on a screen placed perpendicular to the direction of the incident beam. At the first minimum of the diffraction pattern, the phase difference between the rays coming from the two edges of the slit is

(a) zero (b) $\pi/2$ (c) π (d) 2π

[IIT-JEE : 1998]

ln. (d)

At first minima, $b \sin\theta = \lambda$



$\because \sin\theta \simeq \theta$

$\Rightarrow b\theta = \lambda$ or $b\left(\frac{y}{D}\right) = \lambda$ or $y = \frac{\lambda D}{b}$ or $\frac{yb}{D} = \lambda$ ... (i)

Now, at P (First minima) [path difference between the rays reaching from two edges (A and B) will be

$\Delta x = \frac{yb}{D}$ (Compare with $\Delta x = \frac{yd}{D}$ in YDSE)

or $\Delta x = \lambda$ [From equation (i)]

Corresponding phase difference (ϕ) will be

$\phi = \left(\frac{2\pi}{\lambda}\right).\Delta x,\ \phi = \frac{2\pi}{\lambda}.\lambda = 2\pi$

. 4. A beam of light of wavelength 600 nm from a distant source falls on a single slit 1.00 mm wide and the resulting diffraction pattern is observed on a screen 2m away. The distance between the first dark fringe on either side of the central maximum is **[DU 2015]**

(a) 0.6 mm (b) 1.2 mm (c) 1.8 mm (d) 2.4 mm

n. (b) X = 600nm, b = 1.00 mm, D = 2m

Since, we have for single slit

$\because\ b\sin\theta = \lambda$

$$\Rightarrow b\left(\frac{y}{D}\right) = \lambda \Rightarrow y = \frac{\lambda D}{b} = \frac{600\times10^{-9}\,m\times 2m}{1.0\times10^{-3}\,m} = 12\times10^{-4}\,m = 1.2\ mm$$

Ex. 5. Yellow light is used in a single slit diffraction experiment with slit width of 0.6 mm. If yellow light is replaced by *X*-rays, then the observed pattern will reveal

(a) that the central maximum is narrower (b) more number of fringes

(c) less number of fringes (d) no diffraction pattern

[IIT-JEE : 1999]

Soln. **(d)**

Diffraction is obtained when the slit width is of the order of wavelength of light (or any electromagnetic wave) used. Here, wavelength of X-rays (1-100 Å) << slit width (0.6 mm). Therefore, no diffraction pattern will be observed.

10.1.3 White light effect:

The condition for minima of rays incident normally on the slit is, $a\sin\theta = m\lambda$; $\Rightarrow\ \theta = \lambda / a$ [for first order dark band]. As $\lambda_r > \lambda_v$; $\theta_r > \theta_v$. If white light be employed, the central maximum becomes white, while other maxima will be coloured, and red maxima being farther apart than blue.

10.2 FRAUNHOFER DIFFRACTION IN A DOUBLE SLIT:

10.2.1 Introduction:

Let a parallel beam of monochromatic light of wavelength λ be incident normally on a surface containing two slits, each of width a and separated by an opaque feature of width b. The distance between any pair of corresponding points of the two slits is $d = a + b$. According to Huygens-fresnel principle each point of the incident wavefront in the plane of the two slits may be regarded as the source of secondary spherical wavelets. The secondary wavelets travelling normal to the slits are brought to focus by a convex lens *L* on the screen at *C*. The wavelets travelling at an angle θ with the normal are brought to focus at *Q*. Let us calculate the intensity of light at *Q*.

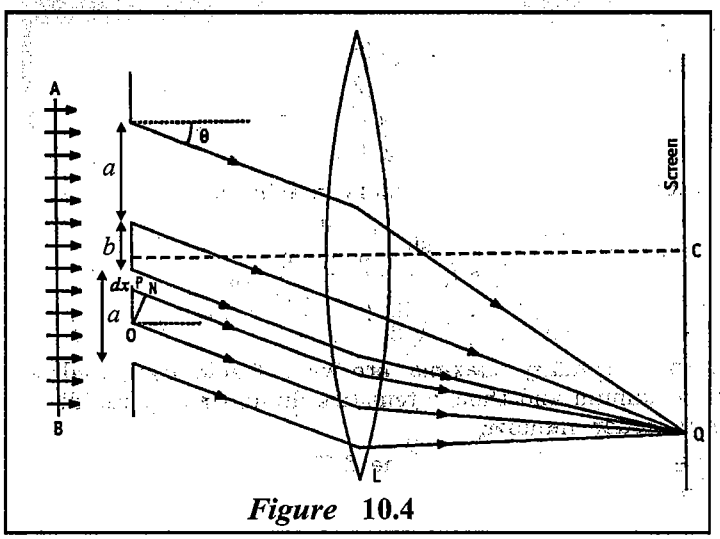


Figure **10.4**

10.2.2 Intensity of the diffraction spots:

Let the complex light disturbance at any instant due to secondary waves from the origin *O* (mid point of the first slit) be represented by $Ae^{i\omega t}$ where *A* is the amplitude and ω is the circular frequency of the wave. The phase difference between the waves at *Q* coming from *O* and from a point *P* at a distance *x* from *O* is given by

$$\frac{2\pi}{\lambda} \times PN = \frac{2\pi}{\lambda} \times x\sin\theta = lx \qquad \ldots (13)$$

where, $l = \dfrac{2\pi \sin\theta}{\lambda}$

Hence the disturbance at Q due to secondary waves from P will be proportional to $e^{i(\omega t - lx)}$. The disturbance at Q due to the diffracting element dx can be written as

$$dy = CAdxe^{i(\omega t - lx)} \qquad \ldots (14)$$

where we assume the amplitude to be proportional to the width dx; C is a proportionality constant. The resultant complex disturbance at Q due to both the slits will be

$$y = \int_{-a/2}^{+a/2} CA.e^{i(\omega t - lx)}dx + \int_{d-a/2}^{d+a/2} CA.e^{i(\omega t - lx)}dx = CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{-a/2}^{+a/2} + CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{d-a/2}^{d+a/2}$$

$$= CAa.\frac{\sin\dfrac{al}{2}}{\dfrac{al}{2}}\left[1 + e^{-ild}\right].e^{i\omega t} \qquad \ldots (15)$$

The resultant intensity I at Q is obtained by multiplying y by its complex conjugate y^*. Thus

$$I = yy^* = (CAa)^2.\frac{\sin^2\dfrac{al}{2}}{\left(\dfrac{al}{2}\right)^2}\left(1 + e^{-ild}\right)\left(1 + e^{ild}\right)$$

$$= (CAa)^2.\frac{\sin^2\dfrac{al}{2}}{\left(\dfrac{al}{2}\right)^2}\left(2 + e^{ild} + e^{-ild}\right) = (CAa)^2.\frac{\sin^2\dfrac{al}{2}}{\left(\dfrac{al}{2}\right)^2}(2 + 2\cos ld)$$

$$= 4I_0\frac{\sin^2\alpha}{\alpha^2}.\cos^2\beta \qquad \ldots (16)$$

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where $I_0 = (CAa)^2$, $\alpha = \dfrac{al}{2} = \dfrac{\pi}{\lambda}.a\sin\theta$ and $\beta = \dfrac{ld}{2} = \dfrac{\pi}{\lambda}(a + b)\sin\theta$

The resultant intensity thus depends on two factors : (i) $I_0\dfrac{\sin^2\alpha}{\alpha^2}$, which gives the diffraction pattern of a single slit and (ii) $\cos^2\beta$, which gives interference pattern of the diffracted light beams from the two slits.

10.2.3 Condition for minima:

The resultant intensity would be zero when either factor of Equation(16) becomes zero.

The factor $\dfrac{\sin^2\alpha}{\alpha^2} = 0$ when $\sin\alpha = 0$ but $\alpha \neq 0$. Therefore, $\alpha = m\pi$; $m = \pm 1, \pm 2, \ldots.$

$$\Rightarrow \qquad a\sin\theta = m\lambda \qquad \ldots (17)$$

These are position of diffraction minima.

The positions of minima due to the factor $\cos^2\beta$ are given by

$$\beta = \pm(2s+1)\pi/2 \; ; s = 0, 1, 2, 3.....$$

$$\Rightarrow \quad (a+b)\sin\theta = \pm(2s+1)\lambda/2 \qquad \ldots (18)$$

These minima are known as the interference minima.

10.2. 4 Condition of maxima:

The positions of maxima due to the factor $\frac{\sin^2\alpha}{\alpha^2}$ are at $\alpha = 0$ and at values approaching $\pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$ etc.

The positions of maxima due to the factor $\cos^2\beta$ given by $\beta = p\pi$; $p\pi = 0, \pm1, \pm2 \pm3, ...$

$$\Rightarrow \quad (a+b)\sin\theta = p\lambda \qquad \ldots (19)$$

10.2.5 Missing order:

If the conditions for maxima of interference pattern equation (19) and minima of diffraction pattern equation(17) are simultaneously satisfied for a given value of θ then the corresponding interference maxima will be missing. If the slit width a is kept constant and separation b is varied then it is obvious from (19) the distance between consecutive interference maxima changes but the diffraction pattern due to single slit remains constant. Let for some value of θ conditions (17) and (19) are satisfied simultaneously. Then,

$$\frac{a+b}{a} = \frac{p}{m} \qquad \ldots (20)$$

Thus for missing order the ratio $\frac{a+b}{a}$ should be the ratio of two integer. For example, if $\frac{a+b}{a} = \frac{2}{1}$; or, $a = b$ then $p = 2m$. Hence 2, 4, 6 etc. orders of interferce maxima are absent which corresponds to 1, 2, 3 etc. order of diffraction dark bands. There will be interference maxima (corresponding to $p = 0, \pm1, \pm2$) within the central diffraction maximum.

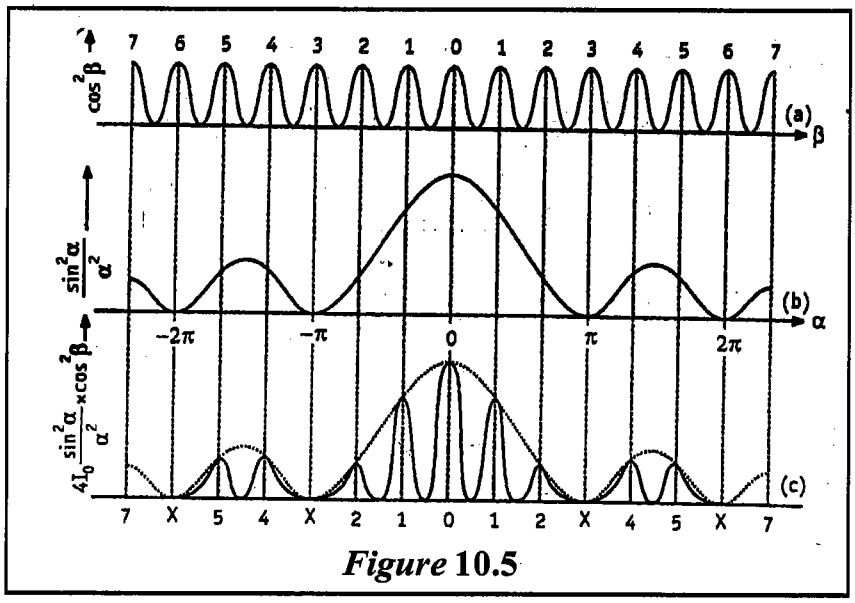


***Figure* 10.5**

Ex. 6. A parallel beam of light is incident normally on a narrow slit of width 0.22 mm. The Fraunhofer diffraction pattern is observed on a screen which is placed at the focal plane of a convex lens whose focal length is 70 cm. Calculate the distance between the first two secondary maxima on the screen. The wavelength of light = 600 nm and the lens is placed very close to the slit.

Soln. The first and second maxima occur at $\alpha = 1.43\,\pi$ and $2.46\,\pi$.

Thus, $a \sin\theta = 1.43\pi$ and 2.46λ

or $\sin\theta = \dfrac{1.43\lambda}{a} = 3.9 \times 10^{-3}$ for 1st maximum

and $\sin\theta = \dfrac{2.46\lambda}{a} = 6.71 \times 10^{-3}$ for 2nd maximum

Consquently the maxima will be separated on the screen by the distance given by

$(6.71 - 3.9) \times 10^{-3} \times 70 = 0.2$ cm.

10.3 PLANE DIFFRACTION GRATING:

10.3.1 Construction:

Plane diffraction grating consists of a number of parallel and equidistant lines ruled on an optically plane and parallel glass plate by a fine diamond point. The number of such ruled lines per mm is of the order of 100. Each ruled line behaves as an opaque line while the transparent portion between two consecutive ruled lines behaves as a slit. If a be the width of a clear space and b be the width of a ruled line, then the distance $(a + b)$ is called grating element or grating constant. The two points in the consecutive clear spaces whose distance of separation is $(a + b)$, are called corresponding points.

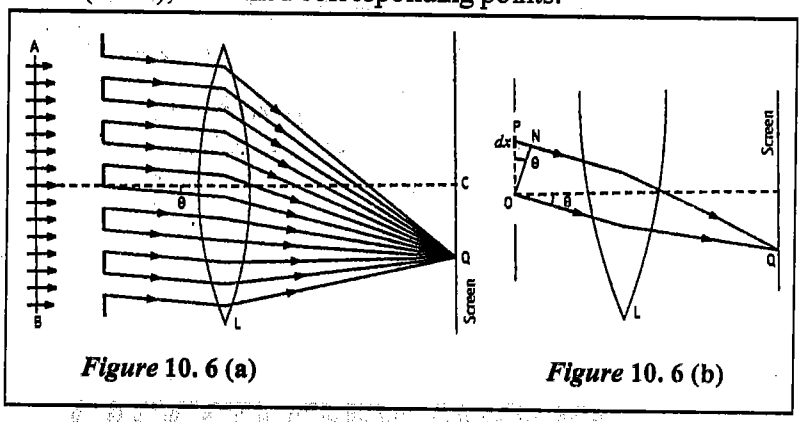


***Figure* 10. 6 (a)** ***Figure* 10. 6 (b)**

Drawing of exactly parallel and equidistant lines on a glass plate by a diamond point is an extremely difficult task and hence the grating becomes too costly. To make it comparatively cheaper, a cast of this ruled surface is made with it some transparent material and such casts are called replica gratings. Cellulose acetate, properly diluted, is poured on the surface of the master grating and then dried to a thin, though film which can be easily separated from the master grating under water. This film can then be mounted on a plane glass plate forming what is known as replica grating. Due to distortion and shrinkages of the seldom functions as well as its master.

10.3.2 Theory:

Let a parallel beam of monochromatic light of wavelength λ be incident normally on a plane diffraction grating consisting of N slits each of width a and with equal opaque space b between two successive slits. According to Huygens-Fresnel principle every point of the incident wavefront in the plane of the slits may be regarded as the origin of secondary spherical wavelets. The secondary wavelets travelling normal to the slits are brought to focus by a convex lens L on the screen at C. The wavelets travelling at an angle θ with the normal are brought to focus at Q. Let us calculate the intensity of light at Q. Let the complex light disturbance at Q due to secondary wavelets from O (the centre of the 1st slit) be represented by $Ae^{i\omega t}$ where A is the amplitude and ω is the circular frequency. The phase difference between the waves at Q [Figure 10.6 (b)] coming from O and from a point P at a distance x from O is given by

$$\frac{2\pi}{\lambda} \times PN = \frac{2\pi}{\lambda} \times x\sin\theta = lx \qquad \ldots (21)$$

where $l = \frac{2\pi}{\lambda}.\sin\theta$

Hence the disturbance at Q due to secondary waves from P will be proportional to $e^{i(\omega t - lx)}$. The disturbance at Q due to the diffracting element dx can be written as

$$dy = CA\,dx.e^{i(\omega t - lx)} \qquad \ldots (22)$$

where we assume the amplitude to be proportional to the width dx; C is a proportionality constant. The resultant complex disturbance at Q due to all the slits will be

$$y = \int_{-a/2}^{+a/2} CA.e^{i(\omega t - lx)}dx + \int_{d-a/2}^{d+a/2} CA.e^{i(\omega t - lx)}dx + \ldots + \int_{(N-1)d-a/2}^{(N-1)d+a/2} CA.e^{i(\omega t - lx)}dx$$

where $d = a + b$ is the grating element.

$$\Rightarrow \quad y = CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{-a/2}^{+a/2} + CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{d-a/2}^{d+a/2} + \ldots + CAe^{i\omega t}\left[\frac{e^{-ilx}}{-il}\right]_{(N-1)d-a/2}^{(N-1)d+a/2}$$

$$= CAa.\frac{\sin\frac{al}{2}}{\frac{al}{2}}\left[1 + e^{-ild} + \ldots + e^{-i(N-1)ld}\right].e^{i\omega t} = CAa\frac{\sin\frac{al}{2}}{\frac{al}{2}}.\frac{e^{iNld} - 1}{e^{-ild} - 1}.e^{i\omega t}$$

The resultant intensity I at Q is obtained by multiplying y with its complex conjugate y^*. Thus

$$I = yy^* = (CAa)^2.\frac{\sin^2\frac{al}{2}}{\left(\frac{al}{2}\right)^2}.\frac{e^{iNld} - 1}{e^{-ild} - 1}.\frac{e^{-iNld} - 1}{e^{+iNld} - 1} = (CAa)^2.\frac{\sin^2\frac{al}{2}}{\left(\frac{al}{2}\right)^2}.\frac{2 - e^{iNld} - e^{-iNld}}{2 - e^{ild} - e^{-ild}}$$

$$= (CAa)^2.\frac{\sin^2\frac{al}{2}}{\left(\frac{al}{2}\right)^2}.\frac{2 - 2\cos Nld}{2 - 2\cos ld} = I_0\frac{\sin^2\alpha}{\alpha^2}\frac{\sin^2 N\beta}{\sin^2\beta} \qquad \ldots (23)$$

CAREER ENDEAVOUR

where $I_0 = (CAa)^2$, $\alpha = \frac{al}{2} = \frac{\pi}{\lambda}a\sin\theta$ and $\beta = \frac{\pi}{\lambda}d\sin\theta = \frac{\pi}{\lambda}(a + b)\sin\theta$

The resultant intensity thus depends on two factors : (i) $I_0\frac{\sin^2\alpha}{\alpha^2} = I_1$ which gives the diffraction pattern of a single slit and (ii) $\frac{\sin^2 N\beta}{\sin^2\beta} = I_2$ which gives the interference pattern of the diffracted light beams from N slits.

10. 3.3 Principal maxima:

If the slit width a is very small and observation is confined to the neighbourhood of the central pattern the variation of the factor $\frac{\sin^2\alpha}{\alpha^2}$ is small and under this condition the maxima will be solely controlled by the

factor $I_2 = \frac{\sin^2 N\beta}{\sin^2 \beta}$. This factor is maximum when $\beta = m\pi$; $m = 0, \pm 1, \pm 2, ...$

$$\Rightarrow \quad (a+b)\sin\theta = m\lambda \qquad \ldots (24)$$

These are known as principal maxima.

For $\beta = m\pi$, $I_2 = \frac{0}{0}$ and hence indeterminate. But in limit $\beta \to m\pi$ we get the maximum value of I_2.

$$\lim_{\beta \to n\pi} \frac{\sin N\beta}{\sin\beta} = \lim_{\beta \to n\pi} \frac{N\cos N\beta}{\cos\beta} = N \qquad \text{[using L Hospital's rule]}$$

This gives $I_2 = N_2$ and $I = I_{pm}$ (say) $= I_0 \frac{\sin^2 \alpha}{\alpha^2} \times N^2 = I_1 \times N^2$ ... (25)

Thus the intensity of principal maxima increases as the number of slits (N) increases, but due to the presence of the factor $\frac{\sin^2 \alpha}{\alpha^2}$, whose value decreases with the increase of the angle of diffraction (θ), the intensity of principal maxima decreases with the increase in the order number of bands.

10.3.4 Conditions for secondary minima and maxima:

The factor $I_2 = \frac{\sin^2 N\beta}{\sin^2 \beta}$ depends on β and for maxima or minima we must have $\frac{dI_2}{d\beta} = 0$.

Now $$\frac{dI_2}{d\beta} = \frac{2N\sin N\beta.\cos N\beta}{\sin^2\beta} - \frac{2\sin^2 N\beta.\cos\beta}{\sin^3\beta}$$

$$= 2\frac{\sin^2 N\beta}{\sin^2\beta}(N\cot N\beta - \cot\beta)$$

Hence for maxima or minima, either $\frac{\sin N\beta}{\sin\beta} = 0$ or, $N\cot N\beta = \cot\beta$ ... (26)

(i) Secondary minima:

When $\sin N\beta = 0$ but $\sin\beta \neq 0$ then $\frac{\sin N\beta}{\sin\beta} = 0$ and hence intensity becomes zero (i.e., minimum).

Thus for minima, $N\beta = \pm s\pi$

$$\Rightarrow \quad (a+b)\sin\theta = \pm\frac{s}{N}\lambda \qquad \ldots (27)$$

where 's' has integral values excepting *0, N, 2N, 3N,* etc. as for these values of s, $\sin\beta = 0$ and we obtain principal maxima. Thus it is evident from Equation (24) and (27) that between two consecutive principal maxima known are $(N-1)$ minima. Hence there will be $N-2$ other maxima known as secondary maxima between any two adjacent principal maxima.

(ii) Secondary maxima

The condition $N\cot N\beta = \cot\beta$ makes $\frac{dI_2}{d\beta} = 0$. Also, it can be shown that this condition makes $\frac{d^2 I_2}{d\beta^2}$ negative. Thus the values of β which satisfy the condition $N\cot N\beta = \cot\beta$ will give the positions of secondary maxima, except $\beta = m\pi$ which gives principal maxima.

Now to find the intensity of secondary maxima we note that $N \cot N\beta = \cot\beta$

$$\Rightarrow \quad N^2 \frac{\cos^2 N\beta}{\cos^2 \beta} = \frac{\sin^2 N\beta}{\sin^2 \beta}$$

$$\Rightarrow \quad \frac{N^2\left(1-\sin^2 N\beta\right)}{1-\sin^2 \beta} = \frac{N^2 \sin^2 N\beta}{N^2 \sin^2 \beta} = \frac{N^2}{1+\left(N^2-1\right)\sin^2 \beta} \qquad \left[\because \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}\right]$$

Hence the intensity of secondary maxima is given by,

$$I_{sm} = I_1 \times \frac{N^2}{1+\left(N^2-1\right)\sin^2} = \frac{I_{pm}}{1+\left(N^2-1\right)\sin^2 \beta}$$

$$\Rightarrow \quad \frac{I_{sm}}{I_{pm}} = \frac{1}{1+\left(N^2-1\right)\sin^2 \beta} \qquad \ldots (28)$$

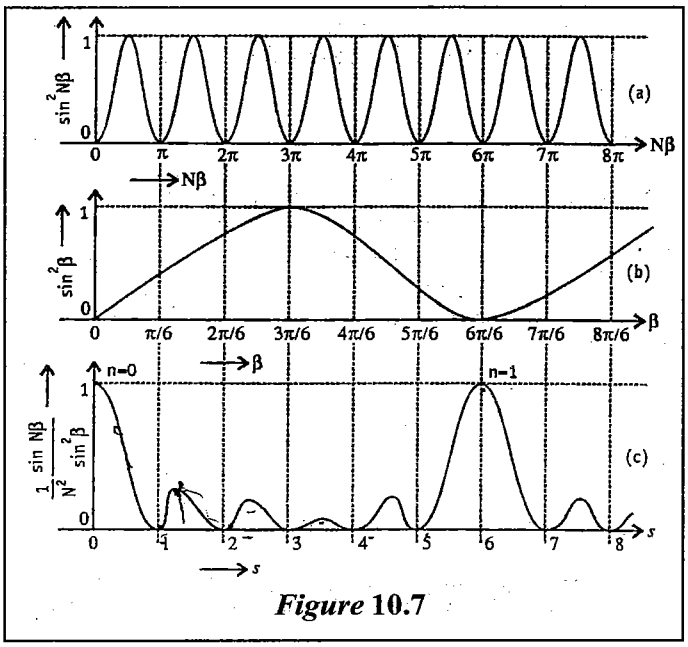


***Figure* 10.7**

10.3.5 Absent spectra and ghosts in a diffraction grating:

For the *m*th order principal maximum in the direction θ, we have the condition

$$(a + b)\sin\theta = m\lambda \qquad \ldots (29)$$

Suppose the value of *a* is such that *s*th order diffraction minimum occurs in the same direction θ then

$$a\sin\theta = s\lambda \qquad \ldots (30)$$

If these two conditions are satisfied simultaneously then *m*th order principal maximum will be absent from the resulting spectra. From equation (29) and (30) we have,

$$\frac{a+b}{a} = \frac{m}{s} \qquad \ldots (31)$$

For example, if $a = b$ then $m = 2s$. Then $m = 2, 4, 6$, etc. order of principal maxima will be absent corresponding to the diffraction minima $s = 1, 2, 3$, etc.

Ghosts

In an ideal grating the rulings should be equally spaced. But in practice there remains some errors in the rulings. If the error is random the grating gives a continuous background illumination. If the error is pro-

gressive in nature the spectral lines becomes sharper in planes which are different from the focal plane of the optical system.

The most common error is periodic in nature. It arises from defects in the driving mechanism of the ruling machine. It gives rise to false lines accompanying the principal maxima of ideal grating. These additional false lines are known as ghosts.

10.3.6 Angular dispersive power of a grating:

The angular dispersive power of a grating is defined as the rate of change of the angle of diffraction (θ) with the change in wavelength i.e. $\frac{d\theta}{d\lambda}$.

If θ is the angle of diffraction for rays of wavelength λ which form the mth order bright band then we have

$$(a+b)\sin\theta = m\lambda$$

$$\Rightarrow \quad \frac{d\theta}{d\lambda} = \frac{m}{(a+b)\cos\theta} \qquad \ldots (32)$$

10.3.7 Comparision of grating and prism spectra:

(i) A prism produces only one spectrum but a grating produces a number of spectra. Consequently prism spectrum is brighter than the grating spectrum.
(ii) In grating spectra there is chance of overlapping of different spectral lines in higher order. As there is only one prismatic spectrum there is no such overlapping in prismatic spectrum.
(iii) In prism the deviation of violet colour is more than the deviation of red colour. But in grating spectrum the deviation of violet is smaller than that of red.
(iv) Resolving power of grating is much greater than that of the prism.
(v) Grating spectrum is nearly normal but the prism spectrum is nearer to normal.
(vi) Dispersion in the grating spectrum does not depend on the material of the grating, but dispersion in prism spectrum depends on the material of the prism.
(vii) Sometimes, due to periodic error in ruling additional false spectra known as ghosts are produced by the grating. But prism spectrum is free from such ghosts.

10.3.8 Determination of wavelength of light by using plane diffraction grating:

Since, $(a+b)\sin\theta = m\lambda \quad \Rightarrow \quad \lambda = \frac{(a+b)\sin\theta}{m} = \frac{\sin\theta}{pm}$

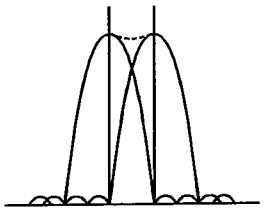
The quantity $p = \frac{1}{a+b}$ represents the number of rulings per unit length, which is usually supplied by the manufacturer of the grating. If not known, it can be easily determined by measuring the angle of diffraction θ for a light of known wavelength and then using the relation $p = \frac{\sin\theta}{m\lambda}$.

10.3.9 Resolution Power:

The resolution power of an optical instrument represent its ability to produce distinctly separate spectral lines of light having two or more close wavelength.

Rayleigh's criterion of resolution

Two spectral lines of equal intensities are just resolved by an optical instrument **when principal maximum of the diffraction pattern due to one falls on the first minimum of the diffraction pattern of the other.**



Just resolved

10.3.10 Resolution Power of a Grating

The resolution power of grating represents its ability to form separate spectral lines for wavelength very close together. It is measured by $\frac{\lambda}{d\lambda}$.

Where $d\lambda$ is the smallest wavelength difference that can be just resolved at wavelength λ.

It can be shown that for a grating resolving power is related with total number of ruling (N) on the grating and the order of spectrum by

$$R = \frac{\lambda}{d\lambda} = Nn$$

Resolving power for central principal maximum is zero as $n = 0$

$$R = Nn = \frac{N(a+b)\sin\theta n}{\lambda}$$

Here, $N(a+b)$ is total width of grating

So, resolving power of grating is directly proportional to total width of the ruled space on the grating.

Ex. 7. The second order maximum in the diffraction of X-rays of 0.20 nonometer wavelength from a simple cubic crystal is found to occur at an angle of thirty degrees with respect to the crystal plane. The distance between the lattice planes is

(a) 1 Angstroms (b) 2 Angstroms (c) 4 Angstroms (d) 8 Angstroms

[JEST : 2012]

Ans. (c)

Ex. 8. A diffraction grating of length 2.5×10^{-2} m is illuminated by a light with two wavelength 5997 A^0 and 6003 A^0. The maximum size of the grating element d (in μm) required to resolve the two wavelengths in the first order is

(a) 50 (b) 25 (c) 75 (d) 100

[IIT-JAM GP : 2012]

Soln. **(b)**

$$\text{Resolving power required} = \frac{\lambda}{d\lambda} = \frac{6000}{6} = 1000$$

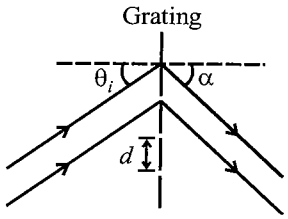
$R = Nn = 1000$

Where n = number of order and N and number of slits

So, $N = 1000$

$$\text{So, grating element } a + d = \frac{\text{length}}{\text{Number of slits}} = \frac{2.5 \times 10^{-2}}{1000} = 25\ \mu\text{m}$$

Ex. 9. A parallel beam of light of wavelength λ is incident on a transmission grating with groove spacing d, at an angle θ_i, as shown in the figure on the left. The plane of incidence is normal to the grooves. After diffraction, the transmitted beam is seen to be an angle α relative to the normal. Which of the following conditions must be satisfied for this to happen ?



(a) $d(\sin\theta_i + \sin\alpha) = n\lambda$ (b) $2d\sin(\theta_i - \alpha) = n\lambda$

(c) $d(\sin\theta_i - \sin\alpha) = n\lambda$ (d) $2d\sin(\alpha + \theta_i) = n\lambda$

[TIFR : 2013]

ns. (a)

x. 10. A narrow beam of light of wavelength 589.3 nm from a sodium lamp is incident normally on a diffraction grating of transmission type. If the grating constant is 1000000 m^{-1}, the number of principal maxima observed in the transmitted light will be

(a) 7 (b) 5 (c) 3 (d) 1

[TIFR : 2010]

ɔln. (d)

Grating constant 10^6 per m.

$$= \frac{1}{10^6} = 10^{-6} m$$

Highest order $n_{max} = \left[\frac{a+b}{\lambda}\right] = \left[\frac{10^{-6}}{589.3\times10^{-9}}\right] = [1.697] = 1 \qquad [\because \sin\theta_{max} = 1]$

x. 11. What is the maximum width, a, of any clear space in a grating having 10^4 lines per inch, if complete second order visible spectrum is to be formed ?

ɔln. $d = \frac{2.54}{10^4} = 2.54\times10^{-4}$ cm.

The maximum angle θ in the second order visible spectrum ($\lambda = 700$ nm) is given by

$$\sin\theta = \frac{2\times7.0\times10^{-5}}{2.54\times10^{-4}} = 0.5512$$

The half angular breadth of the central diffraction band must accommodate the complete second order spectrum, i.e.,

$$\sin\theta = \frac{\lambda}{a} = \frac{7.0\times10^{-5}}{a} = 0.5512 \text{ or } a = \frac{7\times10^{-5}}{0.5512}\text{cm} = 1.27\times10^{-4}\text{ cm.}$$

Note: that this is just half the grating element d.

x. 12. A beam of light is incident normally on a diffraction grating of width 2 cm. It is found that at 30°, n^{th} order diffraction maximum for $\lambda_1 = 5000\mathring{A}$ is super-imposed on the $(n+1)^{th}$ order for $\lambda_2 = 4000\mathring{A}$. How many lines per cm does the grating have? Find out whether the first order spectrum from such a grating can be used to resolve the wavelengths $\lambda_3 = 5800\mathring{A}$ and $\lambda_4 = 5802\mathring{A}$?

[IIT-JAM : 2005]

ln. Condition for m^{th} order maxima is $d\sin\theta = m\lambda$. Thus, $d\sin30^0 = n\lambda_1$

And $d\sin30 = (n+1)\lambda_2$

$\Rightarrow \quad n\lambda_1 = (n+1)\lambda_2 ; \quad n = \dfrac{\lambda_2}{\lambda_1 - \lambda_2} = \dfrac{4000}{1000} = 4$

$\therefore \quad d\sin 30 = 4 \times 5000 \quad \Rightarrow d = 40000\ \mathring{A}$

Number of lines per cm, $N = \dfrac{1\,cm}{d} = \dfrac{1\times10^{-2}}{40000\times10^{-10}} = 2500$

Resolving power of grating = Nm
For $m = 1$
Resolving power = 2500 × 1 = 2500

Resolving power required to resolve to wavelengths λ_3 and λ_4 is given by $\dfrac{(\lambda_3+\lambda_4)/2}{\lambda_4-\lambda_3} = \dfrac{5801}{2} = 2900.5$

Since required resolving power is larger than resolving power of grating, these two wavelengths can not be resolved in first order spectrum.

Ex. 13. A parallel beam of light of diameter 1.8 cm contains two wavelengths 4999.75Å and 5000.25Å. The light is incident perpendicular on a large diffraction grating with 5000 lines per centimeter.
(a) Using Rayleigh criteriion, find the least order at which the two wavelengths are resoleved.
(b) What will be the angular separation (in radians) of the two wavelengths at order n = 2?

[IIT-JAM : 2009]

Soln. (a) Resolving power required to resolve two wavelenghts $= \dfrac{\lambda}{d\lambda} = \dfrac{5000}{0.5} = 10000$

So, resolving power of grating must be greater than 10000.

Resolving power of grating $R = Nn \geq 10000 \Rightarrow n \geq \dfrac{10000}{5000\times1.8} = \dfrac{10}{9} = 1.11;\ n_{min} = 2$

[Total number of line = 5000 lines/cm×1.8 (diameter of parallel beam)

(b) $(a+b)\sin\theta = n\lambda$ ... (1)

Differentiating, we get $(a+b)\cos\theta d\theta = nd\lambda$ and $(a+b) = \dfrac{10^{-2}}{5000} = 2\times10^{-6}m$ ... (2)

From (1), $\sin\theta = \dfrac{n\lambda}{(a+b)} = \dfrac{2\times5000\times10^{-10}}{2\times100} = \dfrac{1}{2};\ \theta = \dfrac{\pi}{6},\ \cos\theta = \dfrac{\sqrt{3}}{2}$

From (2), $2\times10^{-6}\times\dfrac{\sqrt{3}}{2}d\theta = 2\times0.5\times10^{-10}$; $\boxed{d\theta = \dfrac{1}{\sqrt{3}}\times10^{-4}\ \text{radian}}$

Ex. 14. A diffraction grating having N slits, each of width 'b' and period d, is illuminated normally by a monochromatic plane wave of wavelength λ.
(a) Obtain an expression for the highest diffraction order that can be observed. What is the phase difference between waves from first and N^{th} in the highest diffraction.
(b) If alternate slits are covered with a retarder that retards the wave by π, obtain an expression for the intensity distribution of the Fraunhofer diffraction pattern?

[IIT-JAM : 2011]

Soln. (a) Condition for maxima in case of grating $(b+d)\sin\theta = m\lambda$

The highest diffraction order can be observed when $\theta = \dfrac{\pi}{2}$, $(b+d) = \max.\lambda$

The electric field due to 1st slit reaching P, $E_1 = A\frac{\sin\beta}{\beta}\sin(\omega t-\beta+0)$

Due to N^{th} slit, $E_N = A\frac{\sin\beta}{\beta}\sin(\omega t-\beta+(N-1)\phi)$

The phase difference between 1^{st} and N^{th}, $\Delta\phi=(N-1)\phi-0=(N-1)\frac{2\pi}{\lambda}(b+d)\sin\theta$

For highest diffraction order $\theta=\frac{\pi}{2}$. $\Rightarrow \Delta\phi=(N-1)2\pi\frac{(b+d)}{\lambda}$

(b) If alternate slits are covered with retarder which retards wave by π the resultant dielectric field at P is:

$$\vec{E}=\frac{A\sin\beta}{\beta}\left[\sin(\omega t-\beta)+\sin(\omega t-\beta+\phi+\pi)+.....\sin(\omega t-\beta+(N-2)\phi)+\sin(\omega t-\beta)+(N-1)\phi)\right]$$

$$\vec{E}=\frac{A\sin\beta}{\beta}\left[\sin(\omega t-\beta)-\sin(\omega t-\beta+\phi)+.....\sin(\omega t-\beta+(N-2)\phi)-\sin(\omega t-\beta)+(N-1)\phi\right]$$

$$E=\text{imaginary part of }\frac{A\sin\beta}{\beta}\left[e^{i(\omega t-\beta)}-e^{i(\omega t-\beta+\phi)}+.....e^{i(\omega t-\beta+(N-2)\phi)}-e^{i(\omega t-\beta+(N-1)\phi)}\right]$$

$$=\text{Immaginary part of }\frac{A\sin\beta}{\beta}e^{i(\omega t-\beta)}\left[1-e^{i\phi}+e^{i2\phi}-.....+e^{i(N-2)\phi}-e^{i(N-1)\phi}\right]$$

$$=\text{Immaginary part of }\frac{A\sin\beta}{\beta}e^{i(\omega t-\beta)}\left(1-e^{i\phi}\right)\left[1+e^{i2\phi}+....+e^{i(N-2)\phi}\right]$$

$$=\text{Immaginary part of }\frac{A\sin\beta}{\beta}e^{i(\omega t-\beta)}\left(1-e^{i\phi}\right)\left[\frac{1-e^{iN\phi}}{1-e^{i2\phi}}\right]$$

$$=\text{Immaginary part of }\frac{A\sin\beta}{\beta}e^{i(\omega t-\beta)}\left(1-e^{i\phi}\right)\frac{e^{\frac{iN\phi}{2}}}{e^{i\phi}}\left[\frac{e^{\frac{-iN\phi}{2}}-e^{\frac{iN\phi}{2}}}{e^{-i\phi}-e^{i\phi}}\right]$$

$$=\text{Immaginary part of }\frac{A\sin\beta}{\beta}e^{i\left[(\omega t-\beta)+\left(\frac{N}{2}-1\right)\phi\right]}\left(1-e^{i\phi}\right)\frac{\sin N\frac{\phi}{2}}{\sin\phi}$$

$$=\text{Immaginary part of }\frac{A\sin\beta}{\beta}\frac{\sin N\frac{\phi}{2}}{\sin\phi}\left[e^{i\left(\omega t-\beta+\left(\frac{N}{2}-1\right)\phi\right)}-e^{i\left(\omega t-\beta+\frac{N}{2}\phi\right)}\right]$$

$$=A\frac{\sin\beta}{\beta}\frac{\sin N\frac{\phi}{2}}{\sin\phi}\left[\sin\left(\omega t-\beta+\left(\frac{N-1}{2}\right)\phi\right)-\sin\left(\omega t-\beta+\frac{N}{2}\phi\right)\right]$$

$$=A\frac{\sin\beta}{\beta}\frac{\sin N\frac{\phi}{2}}{\sin\phi}2\cos\left(\omega t-\beta+\left(\frac{N-1}{2}\right)\phi\right)\sin\left(\frac{\phi}{4}\right)$$

Resutant intensity. $I=A^2\frac{\sin^2\beta}{\beta^2}\frac{\sin^2 N\gamma}{\sin^2 2\gamma}\sin^2\frac{\gamma}{2}$ $\quad\therefore\quad \gamma=\frac{\phi}{2}$

SOLVED EXAMPLES

1. A convex lens of focal length 40 cm is employed to focus the Fraunhofer diffraction pattern of a single slit of 0.3 mm width. Calculate the linear distance of the first order dark band from the central band. Take the wavelength of light $\lambda = 589$ nm.

Soln. We know that for mth dark band, $a \sin\theta = m\lambda$

Here, $a = 0.3$ mm $= 0.3 \times 10^{-3}$ m; $m = 1$; $\lambda = 589 \times 10^{-9}$ m and $\sin\theta \approx \theta$ radian for θ is very small.

$$\Rightarrow \quad \theta = \frac{589 \times 10^{-9}}{0.3 \times 10^{-3}} \text{ radian} = 0.001963 \text{ radian} = 196.3 \times 10^{-5} \text{ radian}$$

Linear distance $= x = \theta \times f = 196.3 \times 10^{-5} \times 0.4$ m $= 0.7852$ mm

2. Fraunhofer double slit diffraction pattern is observed in the focal plane of a lens of focal length 0.5 m. The wavelength of incident light is 500 nm. The distance between two maxima adjacent to the maximum of zero order is 5 mm and the fourth order maximum is missing. Find the width of each slit and the distance between their centres.

Soln. For missing of 4th order maximum we have $\frac{a+b}{a} = 4$; or, $b = 3a$

Now for 1st order maximum $(a+b)\sin\theta_1 = \lambda$

or, $\quad \theta_1 \approx \frac{\lambda}{a+b} = \frac{\lambda}{4a}$

Therefore, distance between two maxima adjacent to the maximum of zero order is

$$x = f \times 2\theta_1 = \frac{2f\lambda}{4a}$$

$$\Rightarrow \quad a = \frac{f\lambda}{2x} = \frac{0.5 \times 500 \times 10^{-9}}{2 \times 5 \times 10^{-3}} \text{ m} = 0.025 \text{ mm}, \; b = 3a = 0.075 \text{ mm}$$

3. How many orders would be visible, if the wavelength of incident light is 589 nm and the number of lines in the grating is 100 per mm?

CAREER ENDEAVOUR

Soln. For principal maxima, $(a+b)\sin\theta = m\lambda \quad \Rightarrow \sin\theta = pm\lambda$

Here $\theta = 90°$ (maximum); $\lambda = 589 \times 10^{-9}$ m; $p = \frac{1}{a+b} = 100 \times 10^3$ per m

$$\Rightarrow \quad m = \frac{a+b}{\lambda} = \frac{1}{10^5 \times 589 \times 10^{-9}} = 16.32$$

4. Light, which is a mixture of two wavelength 500 nm and 520 nm is incident normally on a plane transmission grating having 1000 lines / mm. A lens of focal length 1 m is used to observe the spectrum on a screen. Calculate the separation of the two lines in the first order spectrum.

Soln. For 1st line, $\sin\theta_1 = pm\lambda = 10^6 \times 1 \times 500 \times 10^{-9} = 0.50$

Thus, $\quad \theta_1 = 30°$

For 2nd line, $\sin\theta_2 = 10^6 \times 1 \times 520 \times 10^{-9} = 0.52$

Thus $\theta_2 = 31°20'$. Hence the angular separation of the two lines is $\theta_2 - \theta_1 = 1°, 20' = (4/3)^0$. That is,

angular separation $= \frac{\pi}{180} \times \frac{4}{3} = \frac{\pi}{135} = 0.02328$ radian. Hence the linear separation of the lines on a screen placed perpendicular to the deflected lines would be $x = 0.02328 \times 1\,\text{m} = 2.328$ cm.

5. The objective of a telescope has a diameter 2.54 m. Assuming the mean wavelength of incident light to be 550 nm calculate the least angular separation of two stars which can be resolved by it.

Soln. The least resolvable angle $= \theta = \frac{1.22\lambda}{D}$ radian.

Here $D = 2.54$ m, $\lambda = 550 \times 10^{-9}$ m.

$$\Rightarrow \quad \theta = \frac{1.22 \times 550 \times 10^{-9}}{2.54} = 2.642 \times 10^{-7} \text{ radian.}$$

$$\Rightarrow \quad \theta = \left(2.642 \times 10^{-7} \times \frac{180}{\pi} \times 3600\right)'' = (0.0545)''$$

5. A diffraction grating 2 cm wide is just able to resolve sodium D-lines in second order. Find the number of rulings per mm.

Soln. We have $\frac{\lambda}{d\lambda} = N.m$

Here m = 2, $\lambda = \frac{589.0 + 589.6}{2}$ nm = 589.3 nm, $d\lambda = 0.6$ nm

Therefore, $$N = \frac{\lambda}{m.d\lambda} = \frac{589.3}{2 \times 0.6} = 491$$

$\therefore$ Number of ruling per mm $= \frac{\text{Total number of lines}}{\text{width of grating in mm}} = \frac{491}{20} = 24.5$

. Consider a grating of width 5 cm with slits of width a = 0.001 mm separated by a distance of 0.002 mm. How many orders would be visible at λ = 550 nm? Calculate the width of principal maxima. Would there be any missing orders?

CAREER ENDEAVOUR

oln. We know, $(a + b)\sin\theta = m\lambda$

Here a = 0.001 mm, b = 0.002 mm, $\lambda = 550 \times 10^{-6}$ mm

For $\theta = 90°$ (maximum), m (order no.) = 5.45

$\therefore$ Maximum number of observable order = 5

Angular width of principal maximum

$$= 2d\theta = \frac{2}{Nm\cot\theta} = \frac{2\lambda}{N(a+b)\cos\theta} = \frac{2 \times 550 \times 10^{-9}}{5 \times 10^{-2}} = 2.2 \times 10^{-5} \text{ radian}$$

where we put, N(a + b) = width of grating $= 5 \times 10^{-2}$ m and $\cos\theta \approx 1$.

Now, $$\frac{a+b}{a} = \frac{3}{1}$$

Hence, 3rd, 6th, 9th ..... etc. order maxima will be missing.

Chapter 11

Polarisation of Light

11.1 INTRODUCTION:

The phenomena of interference and diffraction of light show that the light is a wave motion which may be longitudinal or transverse in nature. It is the phenomenon of polarisation which for its explanation requires that the light must be a transverse wave. In fact in the electromagnetic wave consisting of vibrating electric and magnetic vectors at right angles to each other and also at right angles to the direction of propagation of light.

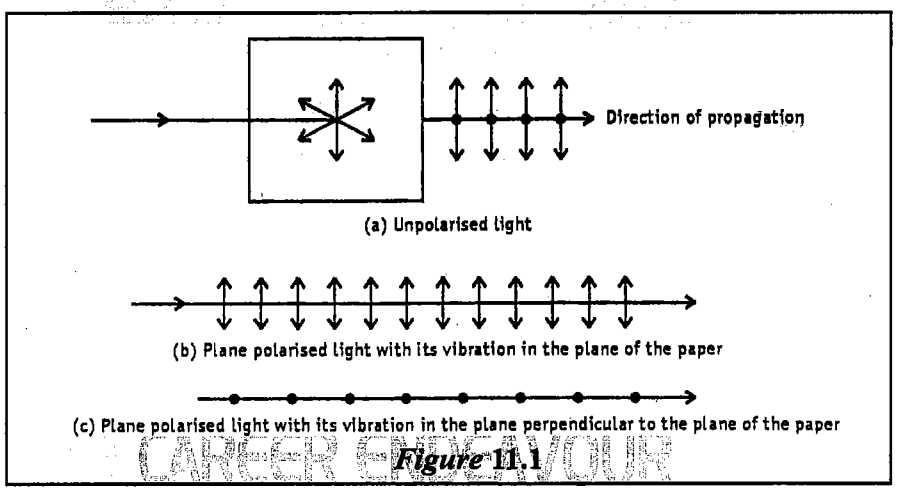

Direction of propagation
(a) Unpolarised light
(b) Plane polarised light with its vibration in the plane of the paper
(c) Plane polarised light with its vibration in the plane perpendicular to the plane of the paper

Figure 11.1

The electric vector is called light vector because it is responsible for the sensation of vision. In an ordinary beam of light with millions of waves the light vectors of component waves will remain in all possible directions on a plane drawn at right angles to the direction of propagation. This happens so due an ordinary beam of light with the electric atoms or molecules in the source. Such an ordinary beam of light with the electric vectors arranges symmetrically about the direction of propagations is called an unpolarised light. We may assume all these vibrations to be resolved into two rectangular components of equal amplitude but having a relative phase difference which varies rapidly and randomly. Now, if by some means one of these rectangular vibrations is cut off we get vibrations of all the component having $\vec{E}$ field in a particular direction. Such a light is said to be plane polarised light (also called p-state light).

The electric vector of a plane polarised light is confined to a particular plane, known as the plane of vibration. This plane contains the $\vec{E}$ -vector and the propagation vector $\vec{k}$. The plane perpendicular to the plane of vibration is known as plane of polarisation.

11.2 POLARISATION BY REFLECTION AND BREWSTER'S LAW:

If an ordinary beam of light is incident on a glass plate at a particular angle $i_p \approx 57°$ it is found that the reflected light becomes almost wholly a plane polarised light, the plane of vibration being perpendicular to

the plane of incidence. For angles other than i_p, the reflected light is partially polarised. This particular angle of incidence at which the reflected light is almost wholly polarised is called the angle of polarisation for the given surface. It depends on the nature of the reflecting medium and the wavelength of light.

Brewster discovered that at the polarising angle the reflected and the refracted rays are just 90° apart. Therefore, the angle of refraction is $90° - i_p$. Hence the refractive index of glass with respect to air is,

$$n = \frac{\sin i_p}{\sin\left(90° - i_p\right)} = \tan i_p \qquad \ldots (1)$$

Thus the tangent of the angle of polarisation is equal to the refractive index of the reflecting medium. This is known as Brewster's law.

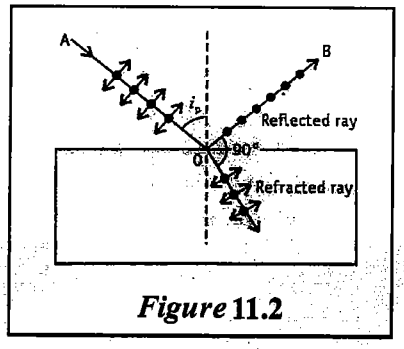


***Figure* 11.2**

ɛ. 1. Parallel light is incident from air on a surface of glass plate a Brewster angle. Which of the following statement is CORRECT ?

(a) Incident and reflected rays are right angle to each other
(b) Incident and refracted rays are parallel to each other
(c) Refracted and reflected rays are at right angle to each other
(d) There is no refracted ray in the glass medium

[IIT-JAM GP : 2007]

ɔln. (c)

ɛ. 2. Unpolarized light is incident at Brewster's angle on the surface of a medium. Which of the following statements is FALSE ?

(a) The parallel component of the light is completely reflected
(b) The reflected light is completely polarized
(c) Some of the incident light is reflected and some is refracted
(d) The reflected light is perpendicular to the refracted light

[IIT-JAM GP : 2009]

ln. (a)

:. 3. Sunlight is incident on the surface of the sea at an angle θ. A sailor viewing through a rotating polaroid can almost block the reflected light by adjusting the angle of the polaroid. The value of θ is about (Refractive index of water is 1.33): **[DU 2014]**

(a) 57° (b) 53° (c) 49° (d) 41°

ln. **(b)** Since the polanoid is able to almost block the reflecting sunlight completely, the reflected sunlight must be lie plane polarized.

From Brewster's law,

$$\mu = \tan\theta$$

$$\Rightarrow 1.33 = \tan\theta$$

$\Rightarrow \quad \theta = \tan^{-1}(1.33) \approx 53^\circ$

where, μ = refractive index of water.

Ex. 4. Longitudinal waves will not show: **[DU 2014]**

(a) Interference (b) Diffraction (c) Polarisation (d) Total internal reflection

Soln. (c) Polarisation is a phenomenon possible only for transverse wave and not logitudinal waves.

Ex. 5. An unpolarized ray of light in a medium with refractive index n_1 is incident on a plane glass surface with refractive index $n_2 (n_2 > n_1)$. The intensity of the reflected light is minimum when the angle of incidence i is given by **[DU 2015]**

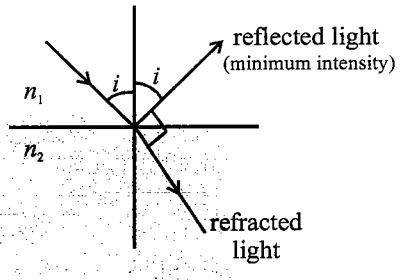
(a) $\tan(i) = n_1 / n_2$ (b) $\tan(i) = n_2 / n_1$

(c) $\tan(i) = 1 / n_1 n_2$ (d) $\tan(i) = n_1 n_2$

Soln. **(b)** $n_1 \sin i = n_2 \sin(90^\circ - i)$

$\Rightarrow \quad \dfrac{n_2}{n_1} = \dfrac{\sin i}{\cos i} = \tan i$

$\Rightarrow \quad \boxed{\tan(i) = \dfrac{n_2}{n_1}}$



11.3 POLARISATION BY REFRACTION:

If a beam of ordinary light is incident on a glass plate at the polarising angle i_p for glass, the transmitted light will be partially plane polarised though the reflected light will be almost wholly plane polarised. If a number of parallel glass plates are used then at each reflection, the rays whose vibrations are perpendicular to the plane of incidence are reflected. The transmitted light contains vibrations in the plane of incidence as well as a reduced amount of vibrations perpendicular to the plane of incidence. Thus with a large number of parallel plates the transmitted light loses more and more of its perpendicular vibrations at each reflection and ultimately becomes wholly polarised whose vibrations remain in the plane of incidence.

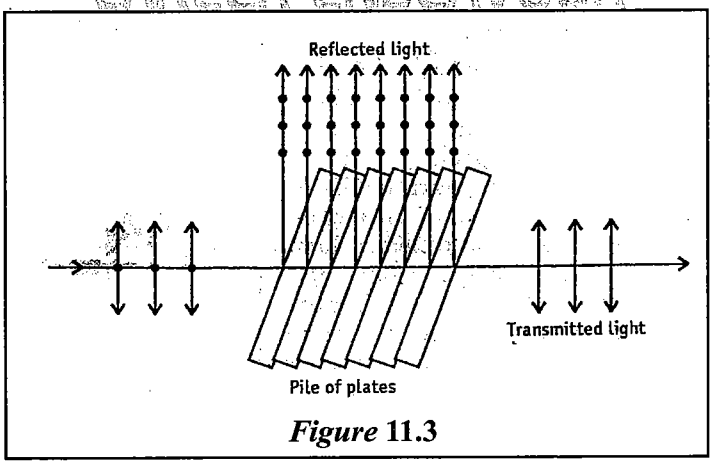


Figure **11.3**

11.4 DOUBLE REFRACTION:

When an ordinary ray of unpolarised light is made incident on a crystal like calcite, quartz etc. it is refracted into two rays. One of these rays obeys the laws of refraction of

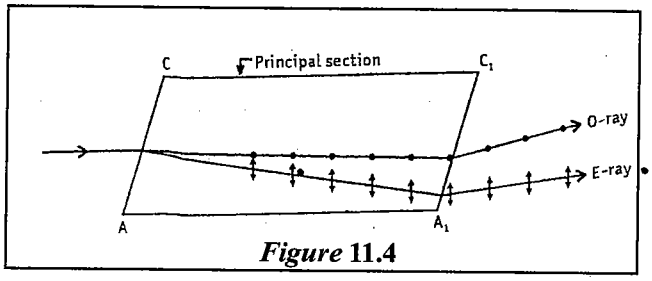


Figure **11.4**

light and is called the ordinary ray (O-ray) while the other ray does not, in general, obey those laws and is called the extraordinary ray (E-ray). Both of these E and O-rays are plane polarised whose vibrations are along and at right angles to the principal section. This phenomenon in which a single incident ray is refracted into two rays is called double refraction or birefringence refracting crystals or birefringent.

11.5 OPTIC AXIS AND PRINCIPAL SECTION OF A CRYSTAL:

Optic axis of crystal is defined as the direction through it, along which if a ray travels then there will be no double refraction of the ray and both the ordinary and extraordinary rays travel with equal velocity along this direction.

A doubly refracting crystal possessing only one optic axis is called a uniaxial crystal.

Principal section of a crystal is its section defined by a plane which passes through the optic axis of the crystal and is perpendicular to its two opposite refracting faces.

11.6 NICOL PRISM:

Nicol prism is an optical device designed from calcite and is used for the production and analysis of plane polarised light. A beam of ordinary light entering the calcite crystal breaks up into E-rays and O-rays by double refraction. The O-ray is cut off by internal refraction while the E-ray is allowed to pass through.

To construct a Nicol prism a calcite rhombohedron of length three times its breadth is produced from natural crystal by cleavage. The end faces are then cut artifically so that the acute angles of the principal section is reduced from 71° to 68°

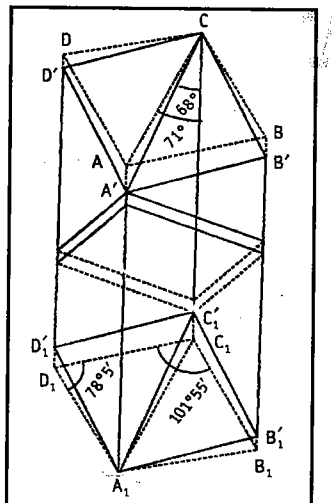


Figure **11.5 (a)**

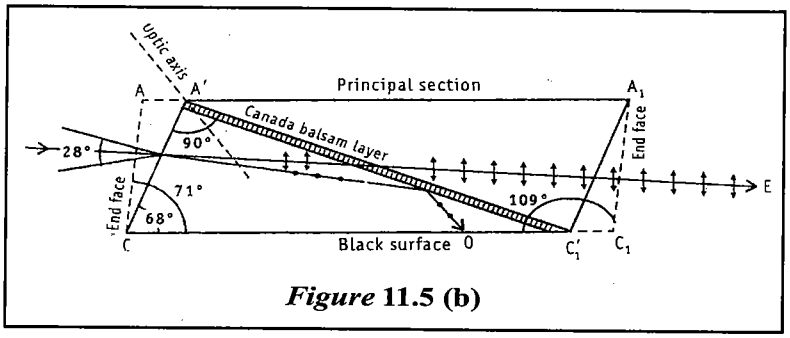


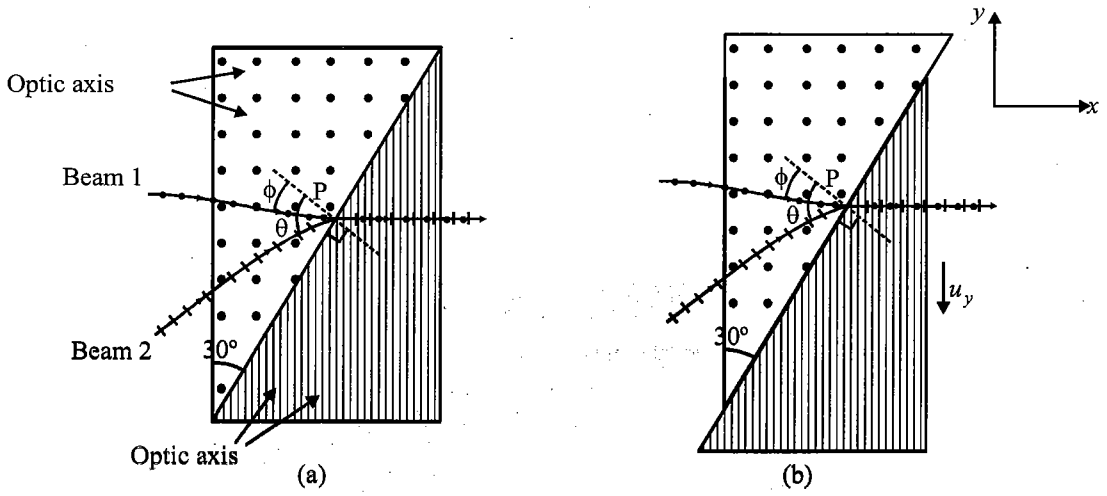
Figure **11.5 (b)**

The resulting crystal is then cut into two portions by a plane perpendicular to both the principal section and the end faces of the rhomb. The cut surfaces are highly polished and cemented together by a layer of Canada-balsam. The sides of the prism are coated with lamp-black and are kept covered by a brass tube. The two end faces are kept open so that the light may be incident on one end face and may emerge out of

the other end face.

A section of the resulting crystal parallel to the principal section is shown in figure 11.5 (b).

Ex. 6. Two orthogonally polarized beams (each of wavelength 0.5 μm and with polarization marked in the figure) are incident on a two-prism assembly and emerge along x-direction, as shown. The prisms are of identical and n_0 and n_e are the refractive indices of the o-ray and e-ray, respectively. Use $\sin\phi = \frac{\sin\theta}{3}$, and $n_0 = \frac{\sqrt{3}+1}{4}$



(a) Find the value of θ and n_e. **[Marks-09]**

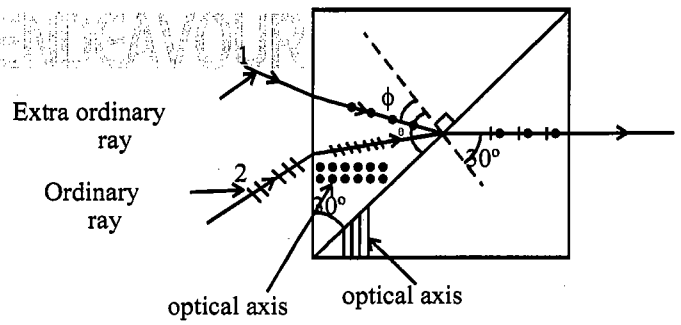
(b) If the right hand side prism starts sliding down with the vertical component of the velocity $u_y = 1\ \mu m/s$, what would be the minimum time after which the state of polarization of the emergent beam would repeat itself? **[IIT-JAM : 2012]**

Soln. For beam 1 (at the interface of two crystals)

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1}$$

$$\frac{\sin\phi}{\sin 30} = \frac{n_o}{n_e} \quad \text{_____ (i)}$$

$$\therefore n_e = \frac{n_o}{2\sin\phi} \quad \text{_____ (ii)}$$



for beam 2 (At the interface of two crystals)

$$\frac{\sin i}{\sin r} = \frac{\mu_2}{\mu_1}; \frac{\sin\theta}{\sin 30} = \frac{n_e}{n_o} \quad \text{_____ (iii)}$$

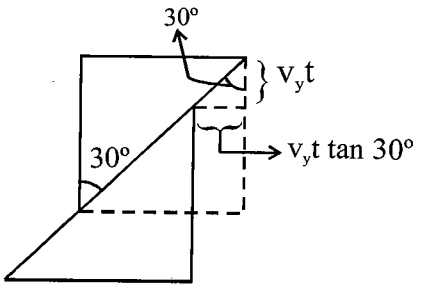
(i) × (ii) gives, $\sin\phi.\sin\theta = \frac{1}{4}$

Since $\sin\phi = \frac{\sin\theta}{3} \Rightarrow \sin\phi\cdot\sin\theta = \frac{\sin^2\theta}{3} = \frac{1}{4} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 60°$

From (ii) $n_e = \dfrac{n_o}{2\times\dfrac{\sin\theta}{3}} = \sqrt{3}\,n_o = \sqrt{3}\left(\dfrac{\sqrt{3}+1}{4}\right) = \dfrac{3+\sqrt{3}}{4}$

(b) State of polarization will repeate itself when path difference between e – ray and o – ray changes by λ

In time t the right crystal moves down by $v_y t$ distance.



Therefore, change in width of medium in time $t = v_y t \tan 30°$

$\therefore$ change in optical path $\Delta x = |n_0 - n_e|\, v_y t \tan 30°$

$\because$ $|n_o - n_e| = \dfrac{3+\sqrt{3}-\sqrt{3}-1}{4} = \dfrac{1}{2}$

$\therefore$ state of polarization will repeate in time t if

$$\Delta x = \frac{1}{2} v_y t \tan 30° = \lambda$$

$$\Rightarrow t = \frac{2\lambda}{v_y \tan 30°} = \frac{2\times 0.5\times 10^{-6}}{1\times 10^{-6}\times\dfrac{1}{\sqrt{3}}} = \sqrt{3}\ \text{Sec.}$$

Due to sliding of right prism, length of medium traversed by the two beams changes, this leads to change in optical path difference.

11.7 MALUS'S LAW:

Malus experimentally obtain a relation which shows how the intensity of light transmitted by the analyser varies with the angle that its plane of transmission makes with that of the polariser. Suppose that the angle between the two planes of transmission be θ at any instant, as shown in Figure 11.**6**. The light vector $AP = E_0$ in the plane polarised light emerging from the polariser may be resolved into two components $AE = E_0 \cos\theta$ and $AO = E_0 \sin\theta$ respectively, along and perpendicular to the plane of transmission of the analyser. The perpendicular component $E_0 \sin\theta$ is eliminated while the parallel component is freely transmitted through the analyser. Hence the intensity of light that emerges from the analyser is given by,

$$I = E_0^2 \cos^2\theta = I_0 \cos^2\theta \qquad \ldots (2)$$

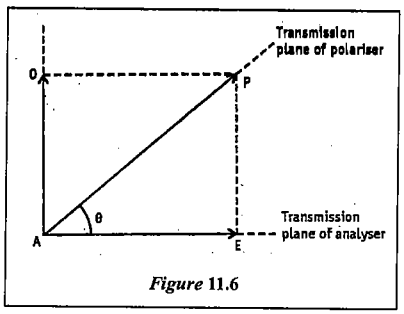


Figure 11.6

where I_0 is the intensity of the polarised light incident on the analyser. This is, of course, one-half of the intensity of the unpolarised light incident on the polariser.

7. Unpolarized light of intensity I_0 passes successively through two identical linear polarisers A and B, placed such that their polarization axes are at an angle of 45° (see figure) with respect to one another.
Assuming A and B to be perfect polarisers (i.e., no absorption losses), the intensity of the transmitted light will

be $I_T =$

(a) $\frac{I_0}{4}$ (b) $\frac{I_0}{2\sqrt{2}}$ (c) $\frac{I_0}{2}$ (d) $\frac{I_0}{\sqrt{2}}$

[TIFR : 2011]

Soln. **(a)**

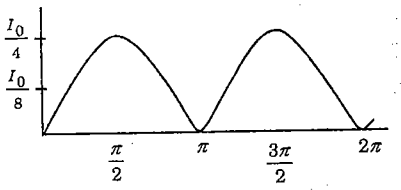
When an unpolarized light of intensity I_0 passes through a polariser intensity of transmitted light is equal to $I_0/2$.

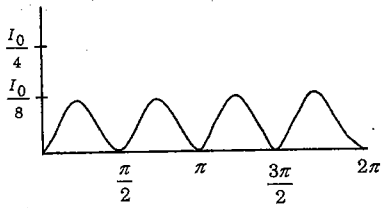
Intensity after $A = \frac{I_0}{2}$

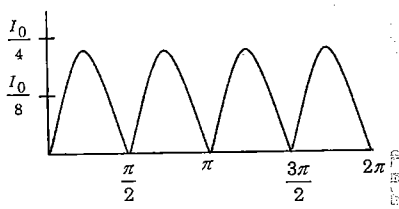
Intensity of light after B

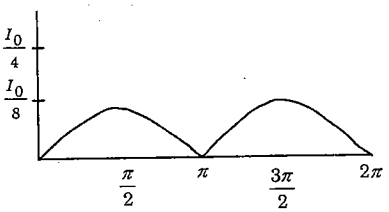
$I = \frac{I_0}{2}\cos^2\theta = \frac{I_0}{4}$

Ex. 8. Three polarizers P, Q and R are placed parallel to each other with their planes perpendicular to the z-axis. Q is placed between P and R. Initially the polarizing directions of P and Q are parallel, but that of R is perpendicular to them. In this arrangement when unpolarized light of intensity I_0 is incident on P, the intensity coming out of R is zero. The polarizer Q is now rotated about the z-axis. As a function of angle of rotation, the intensity of light coming out of R is best represented by

(a) 

(b) 

(c) 

(d) 

[IIT-JAM : 2007]

Soln. **(b)** $I_P = \frac{I_0}{2} \Rightarrow I_Q = I_P\cos^2\theta = \frac{I_0}{2}\cos^2\theta$, and

$$I_R = I_Q\cos^2(90-\theta) \Rightarrow I_R = \frac{I_0}{2}\cos^2\theta\sin^2\theta$$

11.8 SUPERPOSTION OF TWO PLANE POLARISED WAVES VIBRATING IN TWO MUTUALLY PERPENDICULAR PLANES:

Let a plane polarised monochromatic light be incident normally on a calcite plate cut with faces parallel to optic axis (Figure-11.7). Suppose the electric vector makes an angle θ with the optic axis. On entering the crystal the amplitude *'A'* of the incident wave may be assumed to be resolved into two components one along the optic axis forming *E*-wave of amplitude $a = A\cos\theta$ and the other normal to the optic axis forming *O*-wave of amplitude $b = A\sin\theta$. If the incident wave is represented by $E = A\cos(\omega t - kz)$ then the component waves may be represented as

$$x = a\cos(\omega t - kz) \text{ and } y = b\cos(\omega t - kz) \quad \ldots (3)$$

Now the *E*- and *O*-rays travel through the crystal with different velocities ($v_e > v_o$ in calcite) and therefore, they emerge from the crystal with certain relative phase difference, say, δ. So the emergent rays may be represented as

$$\Rightarrow \quad x = a\cos(\omega t - kz + \delta) \qquad \ldots (4)$$

$$\Rightarrow \quad y = b\cos(\omega t - kz) \qquad \ldots (5)$$

$$\Rightarrow \quad \frac{x}{a} = \cos(\omega t - kz)\cos\delta - \sin(\omega t - kz)\sin\delta$$

$$\Rightarrow \quad \frac{x}{a} = \frac{y}{b}.\cos\delta - \sqrt{1 - \frac{y^2}{b^2}}.\sin\delta$$

$$\Rightarrow \quad \left[\frac{x}{a} - \frac{y}{b}\cos\delta\right]^2 = \left(1 - \frac{y^2}{b^2}\right)\sin^2\delta$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{2xy\,.\cos\delta}{ab} + \frac{y^2}{b^2} = \sin^2\delta \qquad \ldots (6)$$

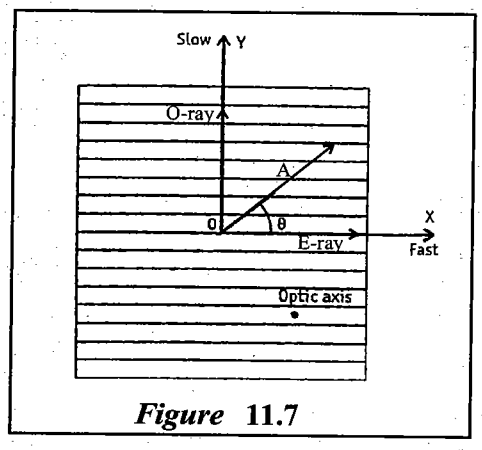


***Figure* 11.7**

This equation in general, represents an ellipse confined within a rectangle of sides $2a$ and $2b$.

11.8.1 Special cases:

(i) If $\delta = 2m\pi$, $m = 0, 1, 2, 3, \ldots$, Equation (6) simplifies to

$$y = \frac{b}{a}x = \tan\theta.x$$

In this case the emergent light is plane polarised, the plane of vibration making an angle θ with the x-axis.

(ii) If $\delta = (2m + 1)\pi$, $m = 0, 1, 2, 3, \ldots$ then Equation (6) simplifies to

$$y = -\frac{b}{a}x = -\tan\theta.x$$

The emergent light is again plane polarised with the direction of vibration making an angle $-\theta$ with the x-axis.

(iii) If $\delta = (2m + 1)\pi/2$, $m = 0, 1, 2, 3, \ldots$ Equation(6) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

In this case the emergent light is elliptically polarised.

If $\delta = \pi/2, 5\pi/2, 9\pi/2$..... etc. then from Equations (4) and (5) we can write

$$x = -a\sin\omega t,\ y = b\cos\omega t \qquad \ldots (7)$$

where we assume $z = 0$ as the plane of observation.

Equation (7) describe an ellipse in counter-clockwise direction (left-handed) with respect to an observer towards whom the wave is travelling. (Note that the vector $\vec{r} = \hat{i}x + \hat{j}y$ rotates counter-clockwise with increasing time).

If $\delta = 3\pi/2, 7\pi/2$,... etc. then from Equations (4) and (5) we get at $z = 0$,

$$x = a\sin\omega t;\ y = b\cos\omega t$$

It describes an ellipse in clockwise direction (right-handed). Here the vector $\vec{r} = \hat{i}x + \hat{j}y$ rotates clockwise with increasing time.

(iv) If $\delta = (2m+1)\pi/2$ and $\theta = 45°$ then a = b and the ellipse reduces to a circle, $x^2 + y^2 = a^2$

The emergent light is then said to be circularly polarised.

As above, for $\delta = \pi/2, 5\pi/2$.... etc. we get left circularly polarised light and $\delta = 3\pi/2, 7\pi/2$,.. etc. we get right circularly polarised light.

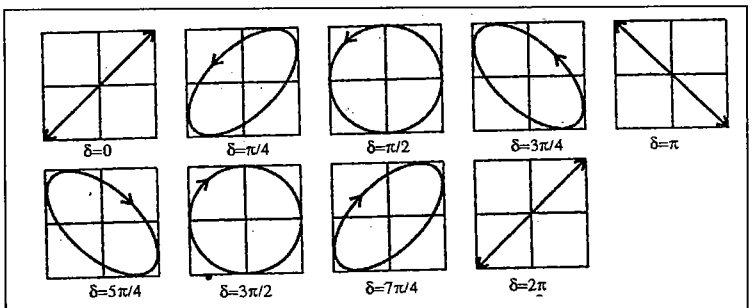


***Figure* 11.8**

11.9 RETARDATION PLATES:

A plate made of doubly refracting crystal with its refracting faces cut parallel to the optic axis and employed to introduce a given phase difference between *E*-ray and *O*-ray is called a retardation plate. If d be the thickness of the plate, n_o and n_e be the refractive indices for *O*-ray and *E*-ray then the path difference introduced by the plate is given by,

$$(n_0 \sim n_e).d \qquad \ldots (8)$$

Corresponding phase difference is given by $\delta = \dfrac{2\pi}{\lambda}(n_0 \sim n_e)d$

QUARTER-WAVE PLATE

If the thickness 'd' of the plate is such that a **path difference of λ/4 or a phase difference of π/2** is introduced between O-ray and E-ray then the plate is called a **quarter-wave plate**. Thus the thickness of a quarter wave plate is given by,

$$(n_o \sim n_c)d = \lambda/4;\quad \Rightarrow d = \frac{\lambda}{4(n_o - n_e)} \qquad \ldots (9)$$

HALF-WAVE PLATE

On the other hand, if the thickness d of the plate is such that a **path difference of λ/2 or a phase difference of π** is introduced between O- and E-waves then the plate is called a **half-wave plate**. The thickness of a half-wave plate is given by,

$$(n_o \sim n_e).d = \lambda/2 \quad \Rightarrow d = \frac{\lambda}{2(n_o \sim n_e)} \qquad \ldots (10)$$

Ex. 9. Consider a beam of light of wavelength λ incident on a system of a polarizer and an analyzer. The analyzer is oriented at 45° to the polarizer. When an optical component is introuduced between them, the output intensity becomes zero. (Light is incident normally on all components). The optical component is:

(a) A full-wave plate
(b) A half-wave plate
(c) A quarter-wave plate
(d) An ordinary glass plate

[IIT-JAM : 2005]

Soln. (b) When half wave plate is introduced, the plane of polarization is rotates by $(\pi - 2\theta)$. Since the intensity of light after analyser is zero. Plane of polarization rotates by an angle of 45°. Then

$$\pi - 2\theta = \frac{\pi}{4}, \quad \theta = \frac{3\pi}{8}$$

Ex. 10. $\vec{E}(x,y,z,t) = A\left(3\hat{i} + 4\hat{j}\right)\exp\left[i(\omega t - kz)\right]$ represents an electromagnetic wave. Possible directions of the fast axis of a quarter wave plate which converts this wave into a circularly polarized wave are

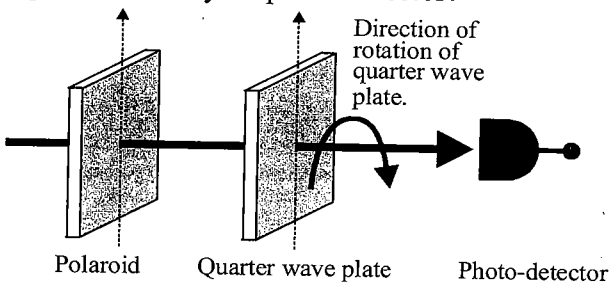
(a) $\frac{1}{\sqrt{2}}\left[7\hat{i} + \hat{j}\right]$ and $\frac{1}{\sqrt{2}}\left[-\hat{i} + 7\hat{j}\right]$
(b) $\frac{1}{\sqrt{2}}\left[3\hat{i} + 4\hat{j}\right]$ and $\frac{1}{\sqrt{2}}\left[4\hat{i} - 3\hat{j}\right]$
(c) $\frac{1}{\sqrt{2}}\left[3\hat{i} - 4\hat{j}\right]$ and $\frac{1}{\sqrt{2}}\left[4\hat{i} + 3\hat{j}\right]$
(d) $\frac{1}{\sqrt{2}}\left[7\hat{i} - \hat{j}\right]$ and $\frac{1}{\sqrt{2}}\left[\hat{i} + 7\hat{j}\right]$

[IIT-JAM : 2006]

Soln. (a) Directions of vibration of $\vec{E}$ wave lie in direction of $\left(3\hat{i} + 4\hat{j}\right)$. Unit vector in this direction is $\hat{n} = \frac{1}{5}\left[3\hat{i} + 4\hat{j}\right]$.

In order to convert the plane light to circularly polarized, fast axis of qualter wave plate must make 45° angle with $\hat{n}$.

Ex. 11. A quarter-wave plate is placed in between a polarizer and a photo-detector. When the optic axis of the quarter-wave plate is kept initially parallel to the pass axis of the polarizer and perpendicular to the direction of light propagation. The intensity of light passing through the quarter-wave plate is measured to be I_0 (see figure). If the quarter wave plate is now rotated by 45° about an axis parallel to the light propagation, what would be the intensity of the emergent light measured by the photo-detector?



(a) $\frac{I_0}{\sqrt{2}}$
(b) $\frac{I_0}{2}$
(c) $\frac{I_0}{2\sqrt{2}}$
(d) I_0

[IIT-JAM : 2010]

Soln. (d)

Ex. 12. A quater wave plate is inserted in between two crossed polarizers. The polarizer pass axes make an angle 45° with the fast and slow axes of the wave plate. If an unpolarized light beam is passed through the system, the transmission would be

(a) 0% (b) 25% (c) 50% (d) 100%

[IIT-JAM GP : 2011]

Soln. **(b)**

After passing through first polarizer, the unpolarized light will become plane polarized with intensity getting half. When plane polarized light passes through half wave plate, it becomes circularly polarized. After passing through the second polarizer, the intensity again becomes half.

So, final intensity is 25% of intensity of unpolarized beam.

Ex. 13. A half wave plate is an anisotropic optical element which introduces a retardation of $\lambda/2$ in the optical path length for the electric field component parallel to its optical axis, where λ is the wavelength of the incident on a half-wave plate with its polarization at 45° to the optical axis, the transmitted light is (compared to the initial polarization)

(a) linearly polarized and rotated by 45° (b) linearly polarized and rotated by 90°

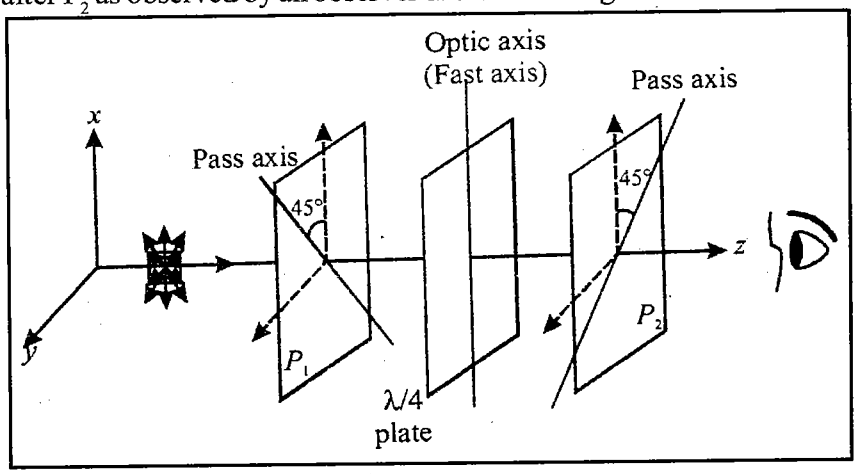
(c) left circularly polarized (d) right circularly polarized

[IISc : 2009]

Soln. **(b)**

When a linearly polarized light with plane of polarization making an angle θ with the optic axis of half wave plate falls on it, the emergent light remains linearly polarized with plane of polarization making an angle of $\pi - \theta$ with the optic axis. So, the amount of rotation is equal to $\pi - 2\theta$.

Ex. 14. A Calcite $\lambda/4$ plate is introduced between the two crossed polarizers (P_1 and P_2) as shown in the figure. An unpolarized beam is incident normal to P_1 from left hand side. Write down the expressions for the electric field associated with the beam after P_1, after $\lambda/4$ plate and after P_2. Also find the state-of-polarization after P_1, after $\lambda/4$ plate and after P_2 as observed by an observer shown in the figure.



[IIT-JAM GP : 2013]

Soln. The polarizer P_1 will produce plane polarized light with the field vibrations along pass axis

So, electric field can be expressed as

$$\vec{E} = \vec{E}_0 \sin(kz - \omega t)$$

Writing component of E_0 in x and y -axis

$$\vec{E} = E_0\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)\sin(kz - \omega t)$$

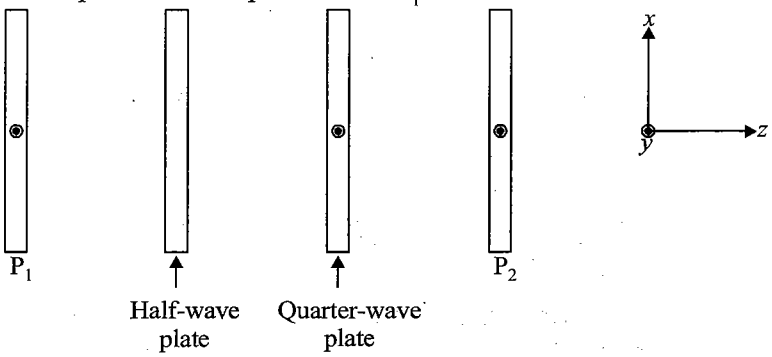
The plane polarized beam of light will fall on the $\lambda/4$ plate with its field vibration making angle of 45° w.r.t. optics axis. So, the $\lambda/4$ plate will form circularly polarized wave the tip of field vector moving in a circle given

by $E_x^2 + E_y^2 = E_1^2$ where $E_1 = \frac{E_0}{\sqrt{2}}$, $E_x = \frac{E_0}{\sqrt{2}} \sin(kz - \omega t)$. The polarizer P_2 will produce plane polarized wave with its field vibration along pass axis and amplitude $\frac{E_0}{\sqrt{2}}$.

$$\vec{E} = \frac{E_0}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}\right)\sin(kz - \omega t)$$

$$= \frac{E_0}{2}\left(\hat{i} - \hat{j}\right)\sin(kz - \omega t).$$

Ex. 15. A half-wave plate and a quarter-wave plate are placed between a polarizer P_1 and an analyzer P_2. All of these are parallel to each other and perpendicular to the direction of propagation of unpolarized incident light (see figure). The optic-axis of the half wave plate makes an angle of 30° with respect to the pass-axis of P_1 and that of the quarter wave plate is parallel to the pass-axis of P_1.



(a) Determine the state of polarization for the light after passing through (i) the half wave plate and (ii) the quarter wave plate.

(b) What should be the orientation of the pass-axis P_2 with respect to that of P_1 such that the intensity of the light emerging from P_2 is maximum?

[IIT-JAM : 2008]

Soln. (a) Let us assume that pass axis of polariser and quarter wave plate is along x-axis.

Let the equation of polarised light be $x = A\sin\omega t$, it will split into two components

$$x' = A\cos\left(\frac{\pi}{6}\right)\sin\omega t = \frac{A\sqrt{3}}{2}\sin\omega t \qquad \text{(Extraordinary ray)}$$

$$y' = A\sin\left(\frac{\pi}{6}\right)\sin\omega t = \frac{A}{2}\sin\omega t \qquad \text{(Ordinary ray)}$$

After passing through half-wave plate. Extra ordinary wave will suffer an additional phase change π with respect to ordinary wave

$$x' = \frac{A\sqrt{3}}{2}\sin(\omega t + \pi) = -\frac{A\sqrt{3}}{2}\sin\omega t \quad \Rightarrow y' = \frac{A}{2}\sin\omega t$$

Equation of lissajous figure, $y' = -\frac{1}{\sqrt{3}}x'$

After passing through half wave plate, the light will remain plane polarises but, the plane of vibration will now shift from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$ with respect to optic axis.

Again, the plane polarises light will go under double refraction and split into ordinary and extraordinary light.

$$y = A\cos\frac{\pi}{6}\sin\omega t \qquad \text{(ordinary light)}$$

$$= A\frac{\sqrt{3}}{2}\sin\omega t$$

$$x = -A\sin\frac{\pi}{6}\sin\omega t \qquad \text{(extraordinary light)}$$

$$= -\frac{A}{2}\sin\omega t$$

The extra-ordinary light will suffer an additional phase

difference of $\frac{\pi}{2}$ with respect to ordinary light.

So, $x = -\frac{A}{2}\sin\left(\omega t + \frac{\pi}{2}\right) = -\frac{A}{2}\sin\omega t \qquad \Rightarrow y = \frac{\sqrt{3}}{2}A\sin\omega t$

Lissagous figure $\frac{x^2}{\left(\frac{A}{2}\right)^2} + \frac{y^2}{\left(\frac{\sqrt{3}}{2}A\right)^2} = 1$ is an ellipse with major axis along y-axis.

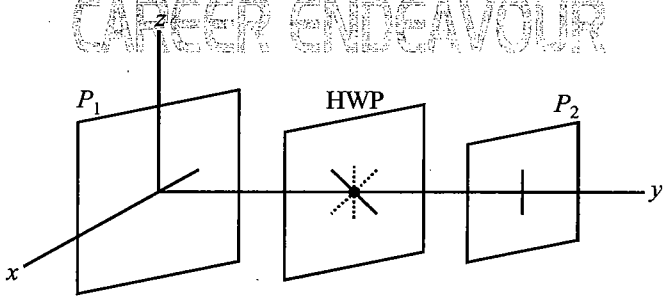
(b) The pass of axis of P_2 should be along the major axis of ellipse i.e. y-axis. So as to get maximum intensity of light.

Ex. 16. A thin rectangular polarizer P_1 with x-axis as pass axis is kept at the origin. Another identical polarizer P_2 with z-axis as the pass is kept parallel to P_1 at $y = d$. A Half Wave Plate (HWP) is introduced between P_1 and P_2 with its optic-axis making 45° with the pass axis of P_1. An unpolarized light propagating along y-axis is incident normally on P_1 from left hand side. What would be the state of polarization after P_1, half wave plate and P_2?

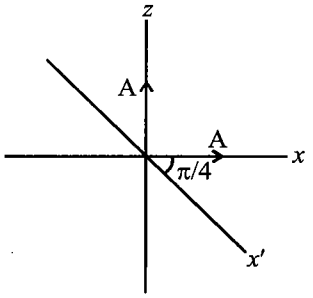
[IIT-JAM GP : 2007]

Soln. When an unpolarized light is incident normally on P_1 the light get plane polarized with vibration only x-axis. So, light will get plane polarized after P_1, when this plane polarized light fall on half wave plate. It breaks into the refracted rays–ordinary and extraordinary ray. After passing

CAREER ENDEAVOUR



through half wave a phase difference of π will be introduced and plane of vibration will change from π/4 to π – π/4 w.r.t. optic axis. So, after half wave plate, the light will remain plane polarized but the plane of vibration which earlier was at angle π/4 w.r.t. optic axis will now be at angle 3π/4 w.r.t. optic axis i.e. along z-direction.



Now, when this plane polarized light falls on P_2 the pass axis of P_2 is parallel to direction of vibration. So, light will completely emerge out of P_2 as plane polarized and without any change in intensity.

11.10 PRODUCTION OF ELLIPTICALLY AND CIRCULARLY POLARISED LIGHT:

11.10.1 Elliptically polarised light:

An elliptically polarised light can be produced by superposing two mutually perpendicular coherent linear vibrations of unequal amplitudes but differing in phase by $\pi/2$. For example, the linear optical vibrations (at $z = 0$), $x = a\cos(\omega t + \pi/2)$, $y = b\cos\omega t$ ... (11)

yield an elliptic vibration given by, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... (12)

Two such linear vibrations can be obtained by allowing a beam of plane polarised light to be incident normally on a quarter-wave plate with the direction of vibration making an angle θ other than 45°with the optic axis of the plate. The incident light of amplitude 'A' can be resolved into two components, one along the optic axis forming E-wave of amplitude $a = A\cos\theta$ and the other normal to the optic axis forming wave of amplitude $b = A\sin\theta$. Let these two components (at $z = 0$) be represented by,

$x = a\cos\omega t$ and $y = b\cos\omega t$. ... (13)

On passing through the $\lambda/4$ plate a relative phase difference of $\pi/2$ will be introduced between these two waves.

11.10.2 Circularly polarised light:

A circularly polarised light can be produced by superposing two mutually perpendicular coherent linear vibrations of equal amplitudes but differing in phase by $\pi/2$. For example, the linear optical vibrations (at $z = 0$),

$x = a\cos(\omega t + \pi/2)$ and $y = a\cos\omega t$ ... (14)

yield a circular optical vibration $x^2 + y^2 = a^2$ ... (15)

Two such linear vibrations can be obtained by allowing a beam of plane polarised light to be incident normally on a $\lambda/4$ plate with the direction of vibration making an angle $\theta = 45°$ to the optic axis of the plate. The incident light of amplitude 'A' can be resolved into two components, one along the optic axis forming E-wave of amplitude $A\cos 45° = A/\sqrt{2}$ and the other normal to the optic axis forming E-wave of amplitude $A\sin 45° = A/\sqrt{2}$. Let these two components (at $z = 0$) be represented as,

$x = \frac{A}{\sqrt{2}}\cos\omega t$ and $y = \frac{A}{\sqrt{2}}\cos\omega t$ ... (16)

Now on passing through the $\lambda/4$ plate a relative phase difference of $\pi/2$ will be introduced between these two waves.

Ex. 17. Which of the following is a circularly polarized standing wave ?

(a) $\Psi(z,t) = (\hat{i}A_1 + \hat{j}A_2)\sin(kz)\cos(\omega t)$

(b) $\Psi(z,t) = (\hat{i}A_1 + \hat{j}A_2)\cos(\omega t - kz)$

(c) $\Psi(z,t) = [\hat{i}\cos(\omega t) + \hat{j}\cos(\omega t - \pi/2)]A\sin(kz)$

(d) $\Psi(z,t) = A\{\hat{i}\cos(\omega t - kz) + \hat{j}\cos[(\omega t - \pi/2) - kz]\}$

[IIT-JAM GP : 2011]

Soln. **(c)**

Ex. 18. Circularly polarized light consists of two perpendicular electromagnetic plane waves of equal amplitude, but differing by a phase θ. The magnitude of θ must be,

(a) $3\pi/4$ (b) π (c) $\pi/2$ (d) $\pi/4$

[IISc : 2006]

Soln. **(c)**

Since, for circular polarization, two waves must have equal amplitude and phase difference must be $\pi/2$.

11.11 DETECTION AND ANALYSIS OF POLARISED LIGHT BY USING NICOL PRISM (OR POLAROID) AND $\lambda/4$ PLATE:

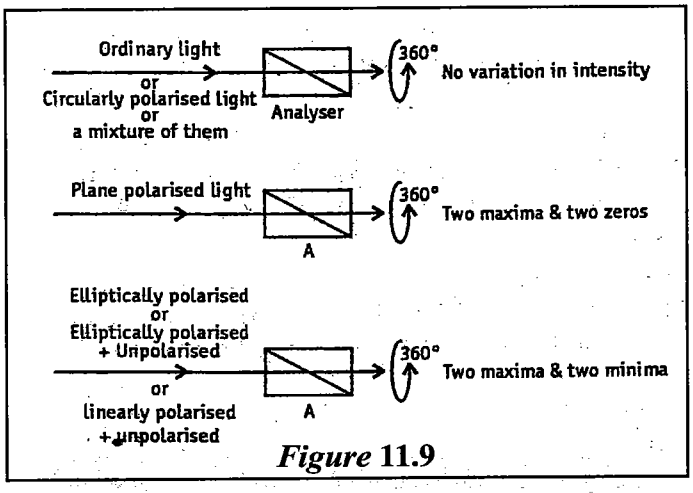


Figure **11.9**

Distinction between circularly polarised light, unpolarised and a mixture of these two:

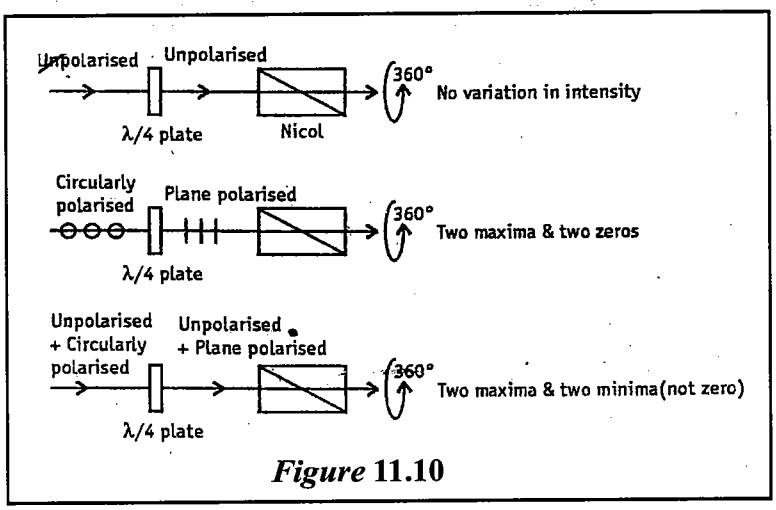


Figure **11.10**

Distinction between elliptically polarised light, mixture of elliptically polarised and unpolarised light and mixture of plane and unpolarised light:

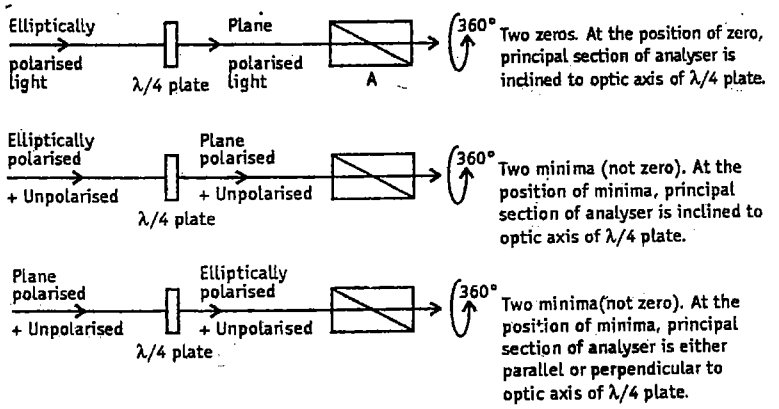
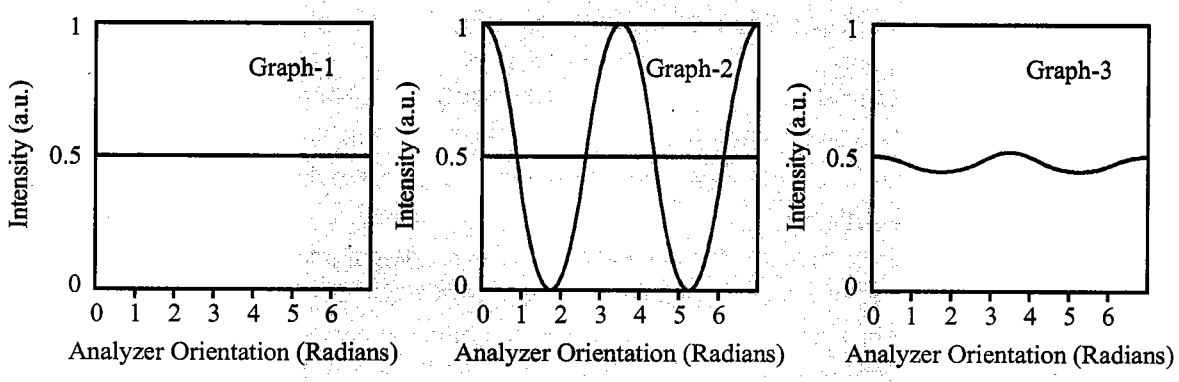

Elliptically polarised light
λ/4 plate
Plane polarised light
A
360°
Two zeros. At the position of zero, principal section of analyser is inclined to optic axis of λ/4 plate.
Elliptically polarised
+ Unpolarised
λ/4 plate
Plane polarised
+ Unpolarised
360°
Two minima (not zero). At the position of minima, principal section of analyser is inclined to optic axis of λ/4 plate.
Plane polarised
+ Unpolarised
λ/4 plate
Elliptically polarised
+ Unpolarised
360°
Two minima(not zero). At the position of minima, principal section of analyser is either parallel or perpendicular to optic axis of λ/4 plate.

***Figure* 11.10**

Ex. 19. Intensity of three different light beams after passing through an analyzer is found to vary as shown in the following graphs. Identify the option giving the correct states of polarization of the incident beams from the graphs.



(a) Graph 1: Linear Polarization, Graph 2: Circular Polarization, Graph 3: Elliptic Polarization
(b) Graph 1: Circular Polarization, Graph 2: Linear Polarization, Graph 3: Elliptic Polarization
(c) Graph 1: Unpolarization, Graph 2: Circular Polarization, Graph 3: Linear Polarization
(d) Graph 1: Unpolarization, Graph 2: Elliptic Polarization, Graph 3: Circular Polarization

[IIT-JAM PH : 2011]

Ans. (b), For explanation, see section 11.11.

Ex. 20. A linearly polarized electromagnetic wave is incident on a quarter wave plate. The emerge wave will in general be **[DU 2015]**

(a) linearly polarized
(b) elliptically polarized
(c) unpolarized
(d) mixture of linearly polarized and unpolarized wave

Soln. (b)

11.11.1 Fast and slow axes of $\lambda/4$ plate:

A quarter-wave plate ($\lambda/4$ plate) has two principal direction, one parallel and another perpendicular to the optic axis which is lying on its surface. The vibrations of the E-ray are along the optic axis while those of O-ray are perpendicular to the optic axis. In the case of a positive crystal like quartz, the *O*-ray travels

faster than *E*-ray in the direction perpendicular to the optic axis and it is customary to call the direction perpendicular to the optic of positive crystal as the fast axis while the axis of positive crystal as the slow axis. In the case of a negative crystal like calcite, *O*-rays travel slower than *E*-ray in the direction perpendicular to the axis and for this the axis perpendicular to the optical axis of a negative crystal is called slow axis while the axis of the negative crystal is called fast axis.

11.12 OPTICAL ACTIVITY:

If plane polarised light is passed through some crystals like quartz along the optic axis the plane of vibration gradually undergoes rotation about the optical axis. The angle of rotation is found to depend on the thickness and nature of the crystal and also on the wavelength of the light employed. This phenomenon which rotates is called optical activity or rotatory polarisation and the substance which rotates the direction of vibration of the incident polarised light is called optically active substance.

There are two classes of active substances. One class of substance rotates the line of vibration of the incident light towards right and the substances belonging to this class are called **dextro-rotatory** substances. Another class of substance rotates the line of vibration towards left and the substances which belong to this class are called **Laevo-rotatory** substances.

11.13 BIOT'S LAWS : MEANING OF SPECIFIC ROTATION, MOLECULAR ROTATION AND ROTATORY DISPERSION:

(1) The angle (θ) of rotation produced by an active substance is proportional to the length (l) of the substance traversed by the ray and is not affected by turning the substance around, consequently the rotation will be annulled if the ray is reflected back through the substance.

(2) The combined rotation produced by two different substanc having different thickness is the algebraic sum of the rotation caused by each separately.

(3) The amount of rotation (θ) in the case of a solution, is proportional to the amount of active substance (m) present per c.c. of the solution.

(4) The amount of rotation varies with the wavelength of the incident light and with temperature of the substance. The rotation is approximately proportional to the inverse square of the wavelength.

In case of a solution, the first and third laws may be combined to have the relation $\theta = s / m$, where s is a constant and is called rotatory power or specific rotation of the solution. Thus specific rotation of a solution is defined as the amount of rotation produced by a 1.0 cm column of liquid containing 1 gm of the active substance per c.c. of the solution. Thus for a solution containing m mg. of active substance per c.c. the rotation (θ) for a path length l cm is given by,

$$\theta = s.m.(l/10) \qquad \ldots (17)$$

where s is the specific rotation. For a pure liquid m is simply the density.

In the case of optically active crystals specific rotation is defined as the rotation produced by a crystal plate of thickness 1 mm.

Molecular rotation is obtained by taking the product of the specific rotation and the molecular weight of the substance. Rotation of a plane polarised light produced by an optically active substance is found to be approximately proportional to the inverse square of the wavelength. So if a plane polarised white light is passed through an optically active substance it will decompose into constituent colours. This phenomenon is known as rotatory dispersion.

11.14 FRESNEL'S EXPLANATION OF ROTATORY POLARISATION

According to Fresnel a plane polarised light may be considered as the superposition of two equal and oppositely circularly polarised light. In case of optically active quartz the secondary wavelets due to O- and E-rays do not touch each other along the optic axis. As a result the two equal and opposite circular vibrations travel along the axis with unequal speed. On passing through the crystal some amount of phase difference (say δ) will be introduced between them. So emergent circular vibrations with plane of vibration

is rotated by certain amount $(\delta/2)$ with respect to the incident vibration. This can be explained as, let vibration at z = 0 be represented by,

$$x = 2a\cos\omega t \qquad \ldots (18)$$

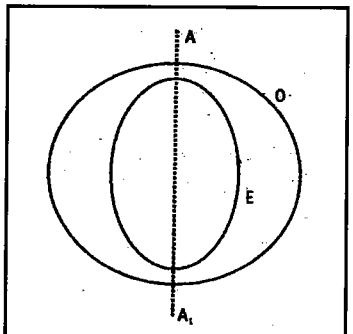


***Figure* 11.11**

This can be resolved into two circular motions:

$$\left.\begin{aligned} x_1 &= a\cos\omega t \\ y_1 &= a\sin\omega t \end{aligned}\right\} \text{ left circular} \qquad \left.\begin{aligned} x_2 &= a\cos\omega t \\ y_2 &= -a\sin\omega t \end{aligned}\right\} \text{ right circular} \ldots (19)$$

These vibration will travel with different velocities and as a result some phase difference will be introduced between them. Let the right circular motion travels faster than the left circular motion and there is a relative phase advancement of right circular motion by δ. Since only relative phases matter the emergent light can be represented by,

$$\left.\begin{aligned} x_1 &= a\cos\omega t \\ y_1 &= a\sin\omega t \end{aligned}\right\} \text{and} \quad \left.\begin{aligned} x_2 &= a\cos(\omega t + \delta) \\ y_2 &= -a\sin(\omega t + \delta) \end{aligned}\right\} \qquad \ldots (20)$$

Thus the resultant vibrations along the x and y directions, on emergence are respectively given by,

$$x = x_1 + x_2 = a\left[\cos\omega t + \cos(\omega t + \delta)\right] = 2a\cos\frac{\delta}{2}.\cos\left(\omega t + \frac{\delta}{2}\right) \ldots (21)$$

CAREER ENDEAVOUR

$$y = y_1 + y_2 = a\left[\sin\omega t - \sin(\omega t + \delta)\right] = -2a\sin\frac{\delta}{2}.\cos\left(\omega t + \frac{\delta}{2}\right) \ldots (22)$$

Taking the ratio of x and y we get

$$y = \tan\left(-\frac{\delta}{2}\right)x$$

So the emergent light is plane polarised whose direction of vibration makes an angle $-\delta/2$ with the x-axis (which is the direction of vibration of incident plane polarised light)

If υ_L and υ_R be respectively the velocities of the left and right handed circular motion within the active substance $(\upsilon_R > \upsilon_L)$ then their time difference $= t = (l/\upsilon_L - l/\upsilon_R)$

[where l = thickness of active substance].

Therefore, Phase difference between the two circular vibrations is given by,

$$\delta = \frac{2\pi}{T}\left(\frac{l}{\upsilon_L} - \frac{l}{\upsilon_R}\right) \qquad \ldots (23)$$

[$\because$ for periodic time difference T, the phase difference is 2π]

$$\Rightarrow \quad \delta = \frac{2\pi l}{cT}\left(\frac{c}{\upsilon_L} - \frac{c}{\upsilon_R}\right) = \frac{2\pi l}{\lambda}(n_L - n_R) \qquad \ldots (24)$$

where n_L and n_R are respectively the refractive indices of the active substance for left and right circularly polarised light along the direction of optic axis.

Again, $$\delta = 2\pi l\left(\frac{1}{\lambda_L} - \frac{1}{\lambda_R}\right) \quad [\because \upsilon_L T = \lambda_L;\ \upsilon_R T = \lambda_R]$$

$$\Rightarrow \quad \delta = \frac{2\pi l \cdot \Delta\lambda}{\lambda^2} \text{ (approx.)} \qquad \ldots (25)$$

Thus the rotation of the plane of vibration is given by

$$\phi = \delta/2 = \frac{\pi l}{\lambda}(n_L - n_R)$$

$$\Rightarrow \quad \phi = \delta/2 = \frac{\pi l \cdot \Delta\lambda}{\lambda^2} \qquad \ldots (26)$$

Thus the rotation varies as the thickness l and inversely as the square of the wavelength λ.

SOLVED EXAMPLES

1. An unpolarised plane light wave of intensity 10 mW/cm^2 passes through two Nicols with their principal sections at 30° to each other. Calculate the intensity of the transmitted wave.

Soln. The intensity (I_1) of the polarised light produced by the 1st Nicol is one half on the incident intensity (I_0). This is so because the incident unpolarised light can be resolved into two orthogonal incoherent linear vibrations each of intensity $I_0/2$ and only one component parallel to the transmission axis will pass through the 1st Nicol.

Thus, $$I_1 = \frac{1}{2}I_0 = 5 \text{ mW/cm}^2$$

Now according to the Malus's law the intensity of light transmitted through the 2nd Nicol is given by,

$$I_2 = I_1 \cos^2\theta = 5 \times \cos^2 30° \text{ mW/cm}^2 = 3.75 \text{ mW/cm}^2$$

2. A plane polarised light of wavelength 600 nm changes to a circularly polarised light on passing through a quartz crystal cut parallel to optic axis. Calculate the minimum thickness to produce such effect. Given $n_e - n_o = 0.005$

Soln. The thickness d should be such that it will introduce a phase difference of $\pi/2$ between E- and O-rays. Thus,

$$\phi = \frac{\pi}{2} = \frac{2\pi}{\lambda} \times (n_e - n_o) \times d \ ; \Rightarrow \ d = \frac{\lambda}{4(n_e - n_o)} = \frac{600 \times 10^{-9}}{4 \times 0.005} \text{ m} = 0.003 \text{ mm}$$

3. A left circularly polarised light is passed through a half-wave plate. What would be the state of polarisation of the emergent light?

or

How can you convert a left-handed circularly polarised light into a right-handed circularly polarised light?

Soln. Let the incident left circularly polarised light be represented by

$$E_x = E_0 \cos(\omega t - kz), \ E\text{-wave}$$

$$E_y = E_0 \sin(\omega t - kz)$$

It represents a right circularly polarised light.

Find the state of polarisation when the x and y components of the electric field are given by

(i) $E_x = E_0 \sin(\omega t + kz)$ $\qquad E_y = E_0 \cos(\omega t + kz)$

(ii) $E_x = E_0 \cos(\omega t + kz)$ $\qquad E_y = \frac{E_0}{\sqrt{2}} \cos(\omega t + kz + \pi)$

oln. (i) $E_x^2 + E_y^2 = E_0^2$. It represent a circular vibration. At any particular position, say, z = 0, the vector $\vec{E} = \hat{i}E_x + \hat{j}E_y = \hat{i}E_0 \sin\omega t + \hat{j}E_0 \cos\omega t$ rotates in clockwise direction. Thus the light under question is a right circularly polarised light.

(ii) $E_y = \frac{E_0}{\sqrt{2}} \cos(\omega t + kz + \pi) = -\frac{1}{\sqrt{2}} E_x$

It represents a linearly polarised light making an angle $\tan^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ with the x-axis.

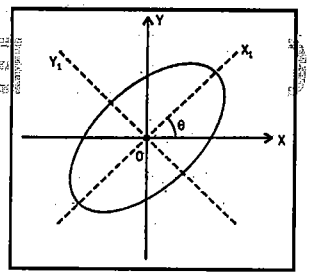
Elliptically polarised light is incident normally on a quarter-wave plate. Find the state of polarisation of the emergent light if the major axis of the quarter-wave plate makes an angle θ with the optic axis (i) θ = 0°, (ii) θ = 90°, (iii) θ = 30°.

oln. Suppose the incident light be a left-handed elliptic vibration represented by $x_1 = a\cos\omega t$, $y_1 = b\sin\omega t$ where the major and minor axis of the elliptic vibration are chosen as X_1 and Y_1 axes respectively. Let the major axis (OX_1) makes an angle θ with the optic axis (OX) of the crystal (see below figure). Hence the elliptic vibration can be resolved into components along OX and OY as,

$$x = x_1 \cos\theta + y_1 \cos(90° + \theta)$$

$$y = x_1 \sin\theta + y_1 \sin(90° + \theta)$$

On passing through the quarter-wave plate a relative phase difference of $\pi/2$



will be introduced between x- and y-components of the electric field. Let the emergent light be represented by

$$\left.\begin{aligned} x &= a\cos(\omega t + \pi/2)\cos\theta + b\sin(\omega t + \pi/2)\cos(90° + \theta), \\ y &= a\cos\omega t \sin\theta + b\sin\omega t \sin(90° + \theta) \end{aligned}\right\} \quad \ldots (1)$$

(i) Now if $\theta = 0$, then $x = -a\sin\omega t$, $y = b\sin\omega t$

$$\Rightarrow \qquad y = -\frac{b}{a}x$$

It represents a linearly polarised light making an angle $\tan^{-1}\left(-\frac{b}{a}\right)$ with the optic (X) axis.

(ii) If $\theta = 90°$ then $x = -b\cos\omega t,\ y = a\cos\omega t$

$$\Rightarrow \quad y = -\frac{a}{b}x$$

The emergent light is again plane polarised with the plane of vibration making an angle $\tan^{-1}\left(-\frac{a}{b}\right)$ with the optic axis.

(iii) If $\theta = 30°$ then from Equation (1)

$$x = -a\sin\omega t.\frac{\sqrt{3}}{2} - b\cos\omega t.\frac{1}{2}$$

$$y = a\cos\omega t.\frac{1}{2} + b\sin\omega t.\frac{\sqrt{3}}{2}$$

Eliminating t from these equation we get,

$$\frac{(bx + ay)^2}{(a^2 - b^2).\left(\frac{1}{2}\right)^2} + \frac{(ax + by)^2}{(b^2 - a^2)\left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

It indicates that the emergent vibration will be elliptic.

6. Write an expression for a linearly polarised light wave of angular frequency ω propagating in the +z direction with its plane of vibration making an angle 60° to the *xz* plane.

Soln. If E_0 be the amplitude of the wave, its x- and y-components are,

$$E_{0x} = E_0\cos 60° = E_0/2 \text{ and } E_{0y} = E_0\sin 60° = \frac{\sqrt{3}}{2}E_0$$

Hence the required expression may be of the form

$$\vec{E}_{(z,t)} = \hat{i}\frac{E_0}{2}\cos(kz - \omega t) + \hat{j}\frac{\sqrt{3}}{2}E_0\cos(kz - \omega t)$$

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7. Describe the state of polarisation of the wave represented by

$$\vec{E}_{(z,t)} = \hat{i}E_0\cos(kz - \omega t) - \hat{j}E_0\sin(kz - \omega t)$$

Soln. The magnitude of $\vec{E}$ is E_0 which is constant, but the direction of $\vec{E}$ is a function of *z* and *t*. Let us examine its behaviour at a fixed space point $z = 0$ and at different times.

$$\vec{E}(0,t) = \hat{i}E_0\cos\omega t + \hat{j}E_0\sin\omega t$$

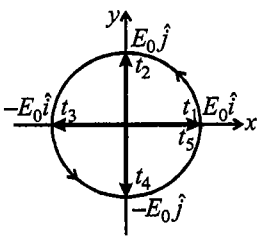
At $t_1 = 0$ $\quad \vec{E} = E_0\hat{i}$

At $t_2 = \frac{T}{4}$ $\quad \vec{E} = +E_0\hat{j}$

At $t_3 = \frac{T}{2}$ $\quad \vec{E} = -E_0\hat{i}$

At $t_4 = \frac{3T}{4}$ $\quad \vec{E} = -E_0\hat{j}$

At $t_5 = T$ $\quad \vec{E} = E_0\hat{i}$



This shows that the electric field vector (E_0) rotates anticlockwise, since the magnitude is constant the tip of $\vec{E}$ sweeps out a circle at a fixed space point.
Thus the wave is left circularly polarised (L-state).

The specific rotation of the quartz for $\lambda = 508.6$ nm is 29.73 deg/mm. Calculate the difference between the refractive indices for left and right circularly polarised light for quartz.

ln. Angle of rotation of the vibration of a plane polarised light by a quartz plate of thickness l for wavelength λ is given by, $\theta = \frac{\delta}{2} = \frac{\pi l}{\lambda}(n_L - n_R)$

$$\Rightarrow \quad n_L - n_R = \frac{\theta.\lambda}{\pi\lambda} = \frac{\left(29.73 \times \frac{\pi}{180}\right) \times 508.6 \times 10^{-9}}{\pi \times 1 \times 10^{-3}} = 8.4 \times 10^{-5}$$

A quartz plate cut with its surface perpendicular to optic axis is required to annul completely the rotation of the plane of polarisation of red light ($\lambda = 760$ nm) by a 26.7 cm length of lactose solution containing 100 g of active substance per litre of the solution. Find the thickness of the plate. Given specific rotation of lactose = $52.5°dm^{-1} . g^{-1} . cm^{-3}$ and for quartz $n_L = 1.53920$ and $n_R = 1.53914$.

n. Rotation produced by quartz of thickness l is

$$\theta = \frac{\delta}{2} = \frac{\pi l}{\lambda}(n_L - n_R) = \frac{\pi l}{760 \times 10^{-9}}(1.53920 - 1.53914) \times \frac{180}{\pi} \text{deg}$$

and rotation produced by the lactose solution $= s.m.\frac{l'}{10} = 52.5 \times \frac{100}{1000} \times \frac{26.7}{10}$ deg

Equating these two rotations we get, $l = 09864 \times 10^{-3}$ m $= 0.9864$ mm

Which of the following phenomenon does NOT produce completely polarized light from unpolarized light?
(a) Absorption (b) Refraction (c) Scattering (d) Reflection

[IIT-JAM GP : 2007]

1. (a)

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The cover page shows the phenomenon of dispersion of light. This proved that the white light is a collection of multiple wavelengths.